6 Jan – 1 Oct, 1897.

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EDITED BY THOMAS CRAIG

WITH THE CO-OPERATION OF SIMON NEWCOMB

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

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JANUARY, 1897.

L. Fuchs

## *Theorie der* [illegible]

Es gibt von [illegible]
cubisch sind, drei [illegible]
Doppelpunkt $C$, 2 [illegible]
haben keine gemeinsam[illegible]
Jene der 1. Art sind [illegible]
mit $C^8$ gemeinsam habe[illegible]
sowie die der 2. Art werden [illegible]
sein. Es bleibt also die Tra[illegible]
sich nur dadurch von ihr unter[illegible]
metrisch zulässigen Weise d[illegible]
sichtlich der Periodicität einz[illegible]
solche Behandlung, da die Au[illegible]
sie führen, allein schon von [illegible]
Construction von hohem Intere[illegible]
meinsten und den speciellsten Fall zu [illegible]
an Umfang der Resultate weitaus üb[illegible]

### *Cap. I.—Die allgemein*[illegible]

1. Die fundamentalen linearen Subst[illegible]
$n$, $x$ die Ordnung und Vielfachheit in $c_6$ für eine [illegible]
$\nu$, $\xi$ Ordnung und Anzahl der Stützpunkte auf $c_6$ f[illegible]

$$n' = 3n - 8x, \quad x' = n - 3x, \qquad (1) \qquad \nu' = 3\nu - [illegible], \quad \xi' = 8\nu - [illegible]$$

* Bezüglich dieses Ausdruckes cf. Cr. J., Bd. [illegible]

# *Theorie der periodischen cubischen Transformationen im Raume $R_3$.*

VON S. KANTOR.

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Es gibt von cubischen Transformationen im $R_3$, die auch im zweiten Raume cubisch sind, drei verschiedene Arten: 1. die $M_2^3$ haben einen gemeinsamen Doppelpunkt $O$, 2. die $M_2^3$ haben eine gemeinsame Doppelgerade $w$, 3. die $M_2^3$ haben keine gemeinsamen Singularitäten und besitzen eine gemeinsame $c_6$, $p=3$. Jene der 1. Art sind Particularfälle von 3. bis auf jene, welche eine Curve $c_6$ mit $O^3$ gemeinsam haben oder eine Degeneration dieser Curve. Diese letzteren sowie die der 2. Art werden systematischer in einer anderen Arbeit zu behandeln sein. Es bleibt also die Transformation der 3. Art nebst sämmtlichen, welche sich nur dadurch von ihr unterscheiden, dass die Curve $c_6$ in irgend einer geometrisch zulässigen Weise degenerirt ist. Diese sind zahlreich und müssen hinsichtlich der Periodicität einzeln behandelt werden, verdienen aber wohl eine solche Behandlung, da die Ausbeute an periodischen Characteristiken, zu denen sie führen, allein schon von fundamentaler Bedeutung und die geometrische Construction von hohem Interesse ist. Das Folgende ist bestimmt, den allgemeinsten und den speciellsten Fall zu erledigen. Wie der letztere den ersteren an Umfang der Resultate weitaus überragt, ist dabei merkwürdig zu verfolgen.

## *Cap. I.—Die allgemeinste cubische Transformation.*

1. Die fundamentalen linearen Substitutionen* derselben sind, wenn mit $n$, $x$ die Ordnung und Vielfachheit in $c_6$ für eine $M_2^4$ bezeichnet werden und mit $\nu$, $\xi$ Ordnung und Anzahl der Stützpunkte auf $c_6$ für eine $M_1^\nu$:

$$n' = 3n - 8x, \quad x' = n - 3x, \qquad (1) \qquad\qquad \nu' = 3\nu - \xi, \quad \xi' = 8\nu - 3\xi. \qquad (2)$$

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* Bezüglich dieses Ausdruckes cf. Cr. J., Bd. CXIV, p. 50.

Die Substitutionen (1) lassen ungeändert die Formen $n^3 - 8x^3$ und $n - 2x$, (2) aber $n^3 - x^3$ und $3n - x$. Die Singularitätencomplexe $n = 4s$, $x = s$ sind die einzigen invarianten durch (1), die Complexe $\nu = s$, $\xi = 2s$ die einzigen invarianten durch (2).

*Theorem I.— Von den Characteristiken der Transformation* $(c_6, c_6')^3$ *ist keine periodisch ausser* $(cc')$ *u. zw. vom Index* 2.

Denn bereits die Verkettung $c_6'$ in $c_6$ liefert: Ebene in $F_3(c')$ in $F_9(c'^3c)$ in $F_{19}(c'^6c^3)$ in $F_{33}(c'^{10}c^6)$ in $F_{51}$ .... und das allgemeine Glied der Ordnungsreihe ist $2s^2 + 1$. Auch ist die char. Function der zugehörigen linearen Substitution $(\rho - 1)^3$.

2. Um die wirklichen Transformationen mit $(cc')$ zu untersuchen, ist nöthig, zu fragen, in wie weit es möglich sei, dass die beiden Fundamentalcurven $c$, $c'$ einer Transformation coincidiren.

Hiezu ist es nützlich, $c_6$ auf eine ebene Curve $C_4$ $p = 3$ eindeutig abzubilden, etwa eben durch eine $T = (c_6, c_6')^3$, deren Fundamental — $c_6$ die $C_4$ in 6 Punkten trifft. Die Trisekanten bilden sich durch jene Tripel ab, welche mit 6 festen Punkten $a_1 \dots a_6$ der $C_4$ die Basis eines Büschels von $C_3$ bilden. Alle Sextupel eines linearen Systemes (Involution) 3. Stufe, $I_6^3$ führen zu demselben Tripelsysteme. Dass ein Punkt 1 in drei Tripeln enthalten ist, beweist man, indem 1 mit $a_1 \dots a_6$ als Fundamentalpunkte einer involutorischen Transformation $\Theta_3(a_1^3 \dots a_6^3\, 1^3)$ genommen werden, worin $C_4$ sich in eine Curve der Ordnung $4.8 - 7.3 = 11$ mit $1^4 a_1^4 \dots a_6^4$ verwandelt, welche $C_4$ in $(4.11 - 7.4 - (6.4 - 7.2)) : 2 = 3$ involutorischen Paaren schneidet. Hieraus entsteht ferner eine eindeutige Beziehung unter den Tripeln und den Punkten von $C_4$. Denn die $C_3$ des Büschels durch 123 schneiden $C_4$ in Punktetripeln, welche mit einem festen Punkte $R$ alineirt sind, dem Gegenpunkte von $123\, a_1 \dots a_6$. Dass 123 durch $R$ eindeutig bestimmt ist, folgt daraus, dass drei mit $R$ alineirte Punkte und $a_1 \dots a_6$ eine einzige $C_3$ bestimmen, welche $C_4$ in 1, 2, 3 schneidet. I. A. kann eine weitere eindeutige Beziehung der Tripel zu den Punkten nicht vorhanden sein, da diese eine ein-eindeutige Correspondenz unter den Punkten bedingen würde. Also:

*Theorem II.— Unter den Punkten und Trisekanten der* $c_6$ *besteht i. A. eine einzige eindeutige Beziehung E, welche einfach aus der Erzeugung einer* $C_4$ *mittelst Stralbüschel und dazu projectivem* $C_3$*-Büschel hervorgeht, wenn die* $C_3$ *gezwungen sind, durch* 6 *feste Punkte von* $C_3$ *zu gehen.*

Seien 123, 12′3′, 12″3″ drei Tripel des Systemes. Dann geht durch 123 1′2′3′, .... je eine $C_3$ und also müssen 2′3′ und 2″3″ oder 23 und 2″3″ oder 23 und 2′3′ je mit dem Gegenpunkte von 123, 12′3′, 12″3″ (und $a_1 \dots a_9$) alineirt und demnach 23, 2′3′, 2″3″ die Seiten eines in $C_4$ eingeschriebenen Dreieckes 456 sein. Man bemerke nun aber, dass 4 der Gegenpunkt von 123 und also derselbe ist, wenn man von 2, 3 ausgehend die beiden anderen je 2, 3 enthaltenden Tripel construirt hätte. Daher:

*Theorem III.—Die eindeutige Beziehung E unter den Punkten und Trisekanten ist auf folgende Weise definirt. Von jedem Stützpunkte einer Trisekante gehen zwei weitere Trisekanten aus, deren Ebene die $C_6$ in einem 6. Punkte schneidet. Dieser Punkt ist für alle drei Ebenen der gleiche und der Trisekante entsprechende Punkt.*

Ein anderer Beweis desselben Theoremes: Es sei $c_6$ Fundamentalcurve von $T$, $P$ ein Punkt von $c_6$, $T_1$, $T_2$, $T_3$ die Ebenen durch je zwei Trisekanten aus $P$. Der $T_1$ entspricht eine $M_2^3$ mit $P_1'' P_2''$ auf der dem $P$ entsprechenden Trisekante von $c_6'$ und die daher die 2 weiteren von $P_1'$, $P_2'$ ausgehenden Paare von Trisekanten $g_1'g_1''$, $g_2'g_2''$ enthält, deren Ebenen $T_1'$, $T_2'$ seien. Jede dieser schneidet $M_2^3$ in einer weiteren Geraden. Wenn aber nicht die Schnittpunkte von $T_1'$ und $c_6'$ mit denen von $T_2'$ und $c_6'$ theilweise zusammenfallen, so müssen die beiden Schnittgeraden in die Schnittlinie von $T_1' T_2'$ fallen. Es müssen also auch die Schnittpunkte von $T_1'$, $T_2'$ mit $c_6'$ zusammenfallen. Ebenso wird die Ebene $T_3'$ der Trisekanten $g_3'g_3''$ des 3. Stützpunktes von $g$ durch diesen Punkt, $P'$, gehen.

3. Auf $C_4$ war aus 1 das Tripel 456 abgeleitet. Aus 2 entsteht auf dieselbe Art ein Tripel, welches ebenfalls 4 enthält und so aus 3. Da aber 4 der Gegenpunkt eines einzigen Tripels 123 ist, können nur drei entstehen und die Tripel 456 bilden also ein System vom 3. Grade. Auch für dieses besteht eindeutige Beziehung zu den Punkten 1, wie man sieht, wenn man die Seiten von 456 mit $C_4$ schneidet. Ueberdies führt das Tripelsystem 456 durch dieselbe Construction wieder zum Tripelsysteme 123 zurück. Denn die 3 Tripel, welche 4 enthalten, haben ihre Gegenseiten vermöge dem Gesagten auf 23, 31, 12 und das von diesen gebildete Dreieck ist also 123. Die beiden Tripelsysteme stehen denn in umkehrbarer Beziehung und es folgt:

*Theorem IV.—Die Ebenen des Trieders der Trisekanten, welche von einem Punkte P der $c_6$ ausgehen, schneiden $c_6$ in 3 Punkten. Diese Tripel bilden ein den Punktetripeln der Trisekanten analoges System und die Beziehung unter P und dem Tripel ist eine eindeutige. Drei Tripel, welche denselben Punkt $P'$ enthalten, entstehen* *aus den Triedern dreier Punkte P, die in einer Trisekante enthalten sind, jener, welche dem P' entspricht.*

In der Ebene folgt ferner, dass sämmtliche Dreiecke des 1. und 2. Systemes eine einzige Curve 12. Cl. umhüllen. Das abgeleitete Tripelsystem entsteht aus den Sextupeln einer $I_6^3$ ganz wie das gegebene. Ich behaupte, dass aus 123 nebst einem mit 4 alineirten Tripel als Sextupel das 2. Tripelsystem, zunächst 456, abgeleitet werden kann. In der That bestimmen 4 und das abgeleitete Tripel eine Gerade, 56 und 23 eine andere und 1 bestimmt $\infty^1$ Geraden. Da aber 123 und das mit 4 alineirte Tripel in einer $C_3$ durch $a_1 \ldots a_6$ sind, so folgt:

*Theorem V.—Die beiden Sextupelsysteme* 3. *Stufe in* $C_4$, *aus welchen die beiden Tripelsysteme entstehen, sind residual für den Schnitt mit Curven* 3. *Ordnung.*

Unter den Tripeln des Systemes ist auch das Schnittpunktetripel von $C_4$ mit $(a_1 \ldots a_5)^2$ und den 5 analogen. Daher: Wenn man durch irgend ein Tripel 123 des Systemes einen Kegelschnitt legt, welcher in 5 Punkten schneidet, so bestimmen diese 5 Punkte mit dem in $E$ entsprechenden Punkte 4 ein Sextupel des Basissystemes. So entstehen alle $\infty^1 . \infty^2 = \infty^3$ Sextupel.* Nimmt man nun $a_1 \ldots a_6$ als Fundamentalpunkte der Abbildung einer $M_2^3$, so wird $c_6$ 6 der Geraden von $M_2^3$ 3-punktig, 6 1-punktig, 15 2-punktig schneiden und es folgt:

*Theorem VI.—Durch* $c_6$ *und eine Trisekante g gehen noch* $\infty^2$ $M_2^3$. *Die g in diesen* $\infty^2$ *Doppelsechsen zugeordneten einfach schneidenden* $\infty^2$ *Geraden gehen sämmtlich durch denselben Punkt von* $c_6$, *welcher jener Trisekante in E entspricht.*

Bemerken wir endlich noch, dass die beiden Sextupelsysteme auf $c_6$ erscheinen als die Schnitte mit $\infty^2$-Systemen von $M_2^3$:†

4. Sechs Trisekanten, welche eine Sechs einer $M_2^3$ durch $c_6$ bilden, werden durch $E$ nicht 6 Punkte in einer Ebene entsprechen. Denn nimmt man $a_1 \ldots a_6$ als Bilder der einfach schneidenden Sechs, so müssten die den Tripeln auf $(a_1 \ldots a_5)^2$ u. analogen entsprechenden 6 Punkte mit $a_1 \ldots a_6$ in einer $C_3$ sein, $C_4$ müsste in $a_1 \ldots a_6$ von einer $C_3$ berührt werden. Die $I_6^3$ müsste mit ihrer residualen übereinstimmen.

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* Die Mannigfaltigkeit der $\infty^2$ Kegelschnitte, welche aus den Sextupeln zu bilden sind, enthält also $\infty^1$ lineare $R_2$.

† In der Abbildung sind auch die Quadrupel erkennbar, zu welchen man durch die Erzeugung der $c_6$ mittelst Systemen von Correlationen gelangt. Die Tetraeder werden abgebildet durch die alineirten Punktquadrupel der $C_4$ und sind definirt durch die Eigenschaft, dass ihre Seitenflächen durch 4 Trisekanten gehen. Jede Ebene durch eine solche schneidet $c_6$ in 3 Punkten, von deren Verbindungslinien jede noch eine Trisekante trifft und mit dieser eine Ebene bestimmt, welche alle drei durch denselben Punkt von $c_6$ laufen.

*Theorem VII.—Wenn eine $c_6$, welche keine* 1—1 *deutige Punkt-Punktcorrespondenz zulässt, als $c_6$ und $c_6'$ einer cubischen Transformation erscheint, ist diese involutorisch und $c_6$ eine Kegelspitzencurve.*

Denn $T$ muss dann $E$ nach beiden Richtungen enthalten. Sie ist aber durch $E$ vollkommen bestimmt, da durch 3 Trisekanten eine $M_2^3$ des $\infty^3$-Systemes geht, welcher die Ebene durch die drei nach $E$ entsprechenden Punkte von $c_6$ wird entsprechen müssen, ist also involutorisch.

In dem $\infty^3$ Systeme von Correlationen wird jedes der gemeinsamen Tetraeder eines Büschels sich selbst conjugirt sein müssen, weshalb alle $\infty^3$ Correlationen Polarsysteme sein müssen. Gleichzeitig folgt:

*Theorem VIII.—Unter den Curven $c_6$ $p=3$ auf einer $M_2^3$ sind Kegelspitzencurven diejenigen, welche ihre einfache Sechs in* 6 *Punkten einer Ebene treffen.*

Um nach dem oben Gesagten in der Abbildung einer $M_2^3$ die $C_4$ zu construiren, welche in $M_2^3$ Kegelspitzencurven liefern, hat man also $C_3$ durch $a_1 \ldots . a_6$ zu construiren, in welchen $a_1$ und der 6. Schnittpunkt mit $(a_1 \ldots . a_6)^2$ contangential sind. Nimmt man den Tangentialpunkt $t$ a priori, so schneidet die $C_4(a_2 \ldots . a_6 t^2)$, Ort der Berührungen aus $t$ an die $C_3$ durch $t$, die $(a_2 \ldots . a_6)^2$ in 3 weiteren Punkten, welche $C_3$ der gewünschten Art liefern. Zu jeder solchen $C_3$ gehören $\infty^3$ sie in $a_1 \ldots . a_6$ berührende $C_4$, welche Kegelspitzencurven in $M_2^3$ liefern.

5. *Theorem IX.—Wenn $c$, $c'$ Fundamentalcurven einer cubischen Transformation sind, so sind sie in eindeutiger Punkt-Punktcorrespondenz, welche nur dann in einer Collineation enthalten ist, wenn $c'$ Kegelspitzencurve ist.*

Denn ein Punkt $P$ von $c$ entspricht eine Trisekante von $c'$ und diese in $E$ einem Punkte $P'$ von $c'$; hierauf ist VII anwendbar.*

*Theorem X.—Wenn $c$ eine Kegelspitzencurve ist, wird auch $c'$ eine Kegelspitzencurve.*

Denn $T$ zusammengesetzt mit der aus VII hervorgehenden Transformation liefert eine Collineation, welche $c$ in $c'$ überführt.

*Corollar.* Wenn $c$ eine Pentaedercurve ist, ist auch $c'$ eine Pentaedercurve. Eine solche hat als Bild eine $C_4$, welcher $\infty^1$ einer $C_2$ umgeschriebene Fünfseite eingeschrieben sind.

---

* Von den $\infty^{24}$ existirenden $c_6$ kann jede nur mit $\infty^{18}$ zu $c$, $c'$ verbunden werden. Die $M_{18}$ geht durch das Element $c_6$ selbst nicht, ausser wenn es der $M_{21}$ angehört, welche die Kegelspitzencurve repräsentirt.

6. Wir sahen oben, dass wenn $I_6^3$ in $C_4$ ein, also nur Berührungssextupel enthält, $c_6$ Kegelspitzencurve ist und es fallen nun das 1. und 2. Tripelsystem zusammen. Also:

*Theorem XI.—Für die Kegelspitzencurve besteht die Eigenschaft, dass die Ebenen der Trisekanten, welche aus einem Punkte $P$ der $c_6$ ausgehen, $c_6$ in 3 Punkten $P_1$, $P_2$, $P_3$ schneiden, welche in einer Trisekante sind u. zw. jener, welche $P$ in $E$ entspricht.*

Werden also von $P$ aus diese Ebenen und die drei aus III gezogen, so erhält man zwei Trieder, deren Seiten sich in $PP_1$, $PP_2$, $PP_3$ schneiden, deren Kantenpaare also Ebenen durch eine Gerade $p$ bestimmen, wo $p$ Polare der Ebene $PP_1P_2P_3$ bezüglich des Kegels mit der Spitze in $P$.

6. Eine eindeutige Correspondenz unter den Punkten von $c_6$ bildet sich auf $C_4$ in eine eindeutige Correspondenz ab, welche also stets in einer Collineation enthalten ist. Soll nun $c_6$ eine $T=(c_6, c_6')^3$ gestatten, so müssen je 6 Trisekanten einer $M_2^3$ durch $c_6$ 6 Punkten einer Ebene entsprechen, die Correspondenz muss $a_1 \ldots . a_6$ (welche durch $E$ den Schnittpunktetripeln der $C_4$ mit $(a_2 \ldots . a_6)^2$ u. analogen entsprechen) in 6 Punkte verwandeln, welche mit $a_1 \ldots . a_6$ in einer $C_2$ sind. Die Collineation in $C_4$ muss also die 1. $I_6^3$ in die 2. und folglich die 2. in die 1. überführen. Also führt sie auch das 1. in das 2. Tripelsystem über und umgekehrt. Daher:

*Theorem XII.—Eine $c_6$, welche nicht Kegelspitzencurve ist, kann nur dann als $(cc')$ für eine cubische Transformation dienen, wenn sie eine Correspondenz enthält, welche die Punktetripel der Trisekanten in die Tripel des Th. IV überführt und reciprok. Sind $\gamma$ solche Correspondenzen* (2. *Art.*) *vorhanden, so gibt es $\gamma$ Transformationen $(cc')^3$ und reciprok.*

Jede Collineation in $c_6$ gibt in $C_4$ eine Correspondenz (1. Art), welche jedes der Tripelsysteme in sich überführt. Zwei Correspondenzen 2. Art geben zusammengesetzt eine Correspondenz (d. i. in $C_4$ eine Collineation) 1. Art. Die Collineation 2. Art ist stets von geradem Index. Für Kegelspitzencurven fallen 1. und 2. Art zusammen, also:

*Theorem XIII.—Eine Kegelspitzencurve gestattet so viele Transformationen $(cc')$, als sie Collineationen in sich gestattet.*

In der That muss die ihr inhärente $T=(cc')^3$ mit irgend einer anderen zusammengesetzt eine Collineation liefern, welche $c$ in sich selbst überführt.

7. Die Frage der Auffindung von $(cc')^3$ ist nun darauf zurückgeführt, $C_4$ mit Collineationen in sich zu finden, welche aber überdies ein Paar residualer $I_6^3$ unter einander vertauschen oder (für XIII) ein System Berührungssextupel in sich transformiren. Die $C_4$ mit Collineationen in sich habe ich Acta Math. Bd. XIX vollständig angegeben und es kann mit der dort II. Theil §2 gegebenen Methode bewiesen werden, dass jede dieser Collineationen ein Paar residualer $I_6^3$ oder eine Berührungs — $I_6^3$ in sich überführt, mit Ausnahme jener vom Index 7.

*Theorem XIV.—Es existiren sowohl allgemeine $c_6$ als Kegelspitzencurven, welche Collineationen vom Index* 2, 3, 4, 6, 9, 12 *gestatten. Die Collineation des Index* 9 *führt aber nur bei Kegelspitzencurven zu einer* $T = (cc')^3$.

8. Es kann auch die Bedingung für die Existenz von endlichen Gruppen von $(cc')^3$ mit gemeinsamer $(cc')$ angegeben werden. Für die Kegelspitzencurve müssen Gruppen von Collineationen gesucht werden, welche eine Berührungs — $T_3^6$ der $C_4$ in sich transformiren, oder nach Uebertragung durch die 1, 2 deutige Transformation, und weil jedes der 56 Systeme $p = 1$, $u = 3$ sich in $C_3$ durch 6 Punkte transformiren lässt, Gruppen von birationalen Transformationen über 6 Punkten $a_1, \ldots . a_6$ welche einen 7. Punkt $a_7$ als gemeinsamen Doppelpunkt besitzen oder ihn in $(a_7^2 a_1, \ldots . a_6)^3$ verwandeln. Diese Gruppen sind aus der anderwärts gegebenen Zusammenstellung zu entnehmen.*

Für die allgemeine $c_6$ macht die erwähnte Uebertragung aus den beiden Sextupelsystemen Schnittpunktesysteme von Curven $C_6 a_1^2 \ldots . a_6^2$ mit der Hesse-schen Curve der $C_3$ durch $a_1 \ldots . a_7$ und die Transformationsgruppen, welche diese unter einander überführen, erweisen sich als dieselben wie vorhin.

Die Transformation $(cc')$ sowie ihre Gruppen werden in der Gesammttheorie als typisch zu bezeichnen sein.

9. Ich bemerke, dass die $M_2^4$ durch $c_6$ und eine Curve $c_9\,(p = q)$, welche $c_6$ in 18 Punkten trifft, ein lineares $\infty^{12}$-System bilden, von dem je 3 sich in 11 Punkten schneiden. Jene hievon, die 9 willkürliche Punkte enthalten, bilden ein $\infty^3$ System des Ranges 2 und können für eine (2, 1) deutige Transformation in einen $R_3'$ verwendet werden. Die Transformationen $(c_6 c_6')$ übertragen sich dann nach $R_3'$ in birationale Transformationen und wenn $c_9$ und die 9 Punkte invariant sind, in Collineationen.

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* Cf. mein Buch: Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene. Berlin, Mayer & Müller, 1895.

10. In jeder involutorischen $(cc')^3$ sind die Schnittcurven $C_3$ der Ebenen mit den entsprechenden $M_2^3$ invariant und für die Ebenen durch einen Doppelpunkt $d_i$ müssen sie also daselbst einen Doppelpunkt haben, weshalb die unendlich nahen Punkte an $d$ alle sich selbst entsprechen, und für Ebenen durch $d_i d_k$ in $d_i d_k$ und einen Kegelschnitt durch $d_i d_k$ zerfallen, sodass $d_i d_k$ sich selbst entspricht und also $c_6$ zweimal schneidet. Also: *Für beide Arten involutorischer* $(cc')^3$ *schneidet* $c_6$ *jede der* 28 *Verbindungsgeraden der* 8 *Doppelpunkte zweimal.*

Jede Correlation des die $(cc')^3$ erzeugenden Systemes hat eine Incidenz $-M_2^3$, die Basispunkte dieses Netzes sind die 8 Doppelpunkte $d_i$. Ferner erzeugen die Ebenenbüschel mit den entsprechenden $M_2^3$-Büscheln $M_2^4$ durch $c_6$ und die $d_i$. Dieses $\infty^4$-System ist quadratisch und daher in einem linearen $\infty^5$-System $R_5$ enthalten, welches auf $d_i d_k$ ein in sich transformirtes lineares Punktepaarsystem ausschneiden muss, in welchem $d_1 d_2$ $\infty^4$-mal als Punktepaar erscheint, so dass es $\infty^5$-mal erscheinen muss. Alle $M_2^4$ des $R_5$ gehen durch die $d_i$. Also:

*Theorem XV.—Durch die* 8 *Doppelpunkte jeder involutorischen* $(cc')^3$ *und die c selbst gehen nicht* $\infty^4$, *sondern* $\infty^5$ $M_2^4$. *Dieselben bilden ein solches System, wo jede* $M_2^4$ *durch irgend einen Punkt p nothwendig durch einen* 2. *Punkt* $p'$ *geht.*

Dieselben sind sämmtlich invariant, da eine quadratische Mannigfaltigkeit unter ihnen nur invariante Elemente enthält. Diese $M_2^4$ sind aber von allen übrigen $\infty^{13}$ wohl unterschieden* und das genügt, um zu schliessen:

*Theorem XVI.—Die durch T in dem* $R_{12}$ *der* $M_2^4$ *durch* $c_6$ *hervorgebrachte collineare Involution hat einen* $R_5$ *und einen* $R_6$ *als Directrixräume.*†

11. Die Curve $c_6$ kann, ohne mehr als 7 scheinbare Doppelpunkte zu erhalten, 1, 2, 3 wirkliche Doppelpunkte annehmen und dadurch $p = 2, 1, 0$ werden. In diesen Fällen kann $c_6$ immer noch Fundamentalcurve der cubischen Transformation sein und es wird auch $c_6'$ 1, 2, 3 wirkliche Doppelpunkte neben 7 scheinbaren Doppelpunkten erhalten. Theorem I gilt auch dann noch. Wie sich die übrigen Theoreme gestalten, auszuführen, würde zu weitläufig sein.

### *Cap. II.—Die cubische Reciprokaltransformation.*

Dieselbe ist verschiedentlich aufgetreten: sei es durch die conjugirten Punktepaare bezüglich eines Netzes von $M_2^2$ mit festem Poltetraeder, sei es durch

* Wenn eine invariante $M_2^4$ 1 bis 5 Punkte $d_i$ enthalten soll, so muss sie dieselben zu Doppelpunkten haben.

† Die sämmtlichen involutorischen Transformationen, welche in dieser Theorie auftreten, sind solche, welche Stralencongruenzen (nicht immer lineare) in sich überführen. Die sei hervorgehoben gegenüber einer allgemeinen (gegentheiligen) unrechtigen Bemerkung eines italienischen Verfassers.

die Polarebenen in Bezug auf ein Tetraeder als $M_2^4$, sei es durch die linearen Systeme linearer Transformationen mit 4 festen Punktepaaren. Eine Erscheinungsform sei noch erwähnt: Die $C_3$ mit festem Polvierseit bilden einen $R_3$, ihre Hesseschen Curven gehen durch die Eckpunkte des Vierseites und die Beziehung unter diesen zwei $R_3$ ist genau eine cubische Reciprokaltransformation, in welcher die 4 dreifach zählenden Geraden und die vier dreieckigen Curven die Hauptelemente sind.*

Die primitivsten Eigenschaften sind: Eine Ebene ist in eine $M_2^3$ mit 4 festen Doppelpunkten $b_1^2 b_2^2 b_3^2 b_4^2$ verwandelt; jede Ebene durch einen der 4 festen Punkte $a_i (i = 1 \ldots . 4)$ entspricht einem Quadrikegel durch $b_i^2 b_k b_l b_m$ in einer ternären quadratischen Transformation $Q^2$, also jeder Ebene durch $a_i a_k$ eine Ebene durch $b_i b_k$ u. zw. $a_i a_k a_l$ der $b_i b_k b_m$. Den Punkten von $a_i a_k$ entspricht die Gerade $b_l b_m$ derart, dass allen Punkten $\bar{p}$, unendlich nahe einem Punkte $p$ von $a_i k_k$, die Punkte $\bar{q}$ unendlich nahe an $b_l b_m$ und in einer bestimmten Ebene durch $b_l b_m$, nämlich jener, welche der $a_l a_m q$ entspricht, zugehören.†

Unter den Nachbarpunkten von $a_i$ und den Punkten von $b_k b_l b_m$ besteht die erwähnte $Q^2$; einer Geraden entspricht eine $C_3$ durch $b_1 b_2 b_3 b_4$ mit durch jene Transformationen $Q^2$ bestimmten Tangenten. Das Enthalten von $a_i$ vermindert die Ordnung der entsprechenden Curve um 2, zwei entsprechende Curven schneiden $a_i a_k$ und $b_i b_k$ in gleich vielen Punkten ausserhalb $a_i$, $a_k$; $b_i$, $b_k$. Hieraus folgen nun die folgenden Theoreme:

*Theorem I.—Die fundamentalen linearen Substitutionen für die Verwandlung der $M_2$ durch die Reciprokaltransformation sind*

$$
\begin{aligned}
n' &= 3n - x_1 - x_2 - x_3 - x_4, \\
x_1' &= 2n - x_2 - x_3 - x_4, \\
x_2' &= 2n - x_1 - x_3 - x_4, \\
x_3' &= 2n - x_1 - x_2 - x_4, \\
x_4' &= 2n - x_1 - x_2 - x_3, \\
y_{34}' &= n - x_1 - x_2 + y_{12}, \\
y_{24}' &= n - x_1 - x_3 + y_{13}, \\
y_{23}' &= n - x_1 - x_4 + y_{14}, \\
y_{12}' &= n - x_3 - x_4 + y_{34}, \\
y_{13}' &= n - x_2 - x_4 + y_{24}, \\
y_{14}' &= n - x_2 - x_3 + y_{23}.
\end{aligned}
\tag{1}
$$

* Dies ist sofort auf den $R_r$ zu verallgemeinern.

† Die hier folgende Theorie hatte ich bis inclusive §8 vollständig im Jahre 1884 abgefasst, nur die Theoreme I bis XIII rühren aus dem Januar 1888 her.

Hier bezeichnen $y_{ik}$ die Vielfachheiten, welche die Kanten $a_i a_k$ für die $M_2^n(a_1^{x_1} a_2^{x_2} a_3^{x_3} a_4^{x_4})$ besitzen, entsprechend $y'_{ik}$ für $M_2^{n'}$. Die letzten 6 Zeilen beweisen sich so: Eine Gerade über $b_i b_k$ schneidet $M'$ in $n' - y'_{ik}$ Punkten und die entsprechende $O_2$ schneidet $M$ in $2n - x_i - x_k - y_{lm}$ Punkten, also $y'_{ik} = n' - 2n + x_i + x_k + y_{lm} = n - x_i - x_k + y_{lm}$.

*Theorem II.—Die fundamentalen linearen Substitutionen für die Verwandlung der $M_1$ durch die Reciprokaltransformationen sind*

$$\begin{aligned}
\mathfrak{n}' &= 3\mathfrak{n} - 2\mathfrak{x}_1 - 2\mathfrak{x}_2 - 2\mathfrak{x}_3 - 2\mathfrak{x}_4 - \mathfrak{y}_{12} - \mathfrak{y}_{13} - \mathfrak{y}_{14} - \mathfrak{y}_{34} - \mathfrak{y}_{42} - \mathfrak{y}_{23},\\
\mathfrak{x}'_1 &= \mathfrak{n} - \mathfrak{x}_2 - \mathfrak{x}_3 - \mathfrak{x}_4 - \mathfrak{y}_{34} - \mathfrak{y}_{42} - \mathfrak{y}_{23},\\
\mathfrak{x}'_2 &= \mathfrak{n} - \mathfrak{x}_1 - \mathfrak{x}_3 - \mathfrak{x}_4 - \mathfrak{y}_{13} - \mathfrak{y}_{14} - \mathfrak{y}_{34},\\
\mathfrak{x}'_3 &= \mathfrak{n} - \mathfrak{x}_1 - \mathfrak{x}_2 - \mathfrak{x}_4 - \mathfrak{y}_{12} - \mathfrak{y}_{14} - \mathfrak{y}_{42},\\
\mathfrak{x}'_4 &= \mathfrak{n} - \mathfrak{x}_1 - \mathfrak{x}_2 - \mathfrak{x}_3 - \mathfrak{y}_{12} - \mathfrak{y}_{13} - \mathfrak{y}_{23},\\
\mathfrak{y}'_{34} &= \mathfrak{y}_{12},\\
\mathfrak{y}'_{24} &= \mathfrak{y}_{13},\\
\mathfrak{y}'_{23} &= \mathfrak{y}_{14},\\
\mathfrak{y}'_{12} &= \mathfrak{y}_{34},\\
\mathfrak{y}'_{13} &= \mathfrak{y}_{24},\\
\mathfrak{y}'_{14} &= \mathfrak{y}_{23},
\end{aligned}$$

Hier bezeichnen $\mathfrak{y}_{ik}$ die Anzahlen der Punkte, welche $M_1^{\mathfrak{n}}(a_1^{\mathfrak{x}_1} a_2^{\mathfrak{x}_2} a_3^{\mathfrak{x}_3} a_4^{\mathfrak{x}_4})$ auf der Kante $a_i a_k$ besitzt, entsprechend $\mathfrak{y}'_{ik}$ für $M_1^{\mathfrak{n}'}$. Die Werthe von $\mathfrak{x}'_i$ ergeben sich aus den drei $Q^2$ von oben, die letzten Zeilen aus Schnittebenen durch $a_i a_k$.

Ich werde die Substitutionen (1), (2), wenn die letzten 6 Zeilen fehlen, als die unvollständigen Substitutionen (1), (2) bezeichnen. Die Reciprokaltransformation wird als $(a_i\,;\,b_i)^3$ oder $(a\,;\,b)^3$ angeführt werden.

*Theorem III.—Die bilineare Form*

$$n.\mathfrak{n} - \sum_1^4 x_i \mathfrak{x}_i - \sum_{i,k} y_{ik} \mathfrak{y}_{ik} \tag{3}$$

*bleibt ungeändert, wenn man die* $n$, $x$, $y$ *durch* (1), *die* $\mathfrak{n}$, $\mathfrak{x}$, $\mathfrak{y}$ *durch* (2) *transformirt.*

Denn diese Form stellt die Anzahl der Schnittpunkte einer $M_2^n$ mit einer $M_1^{\mathfrak{n}}$ vor und diese Anzahl bleibt durch $(a_i\,;\,b_i)^3$, also (3) durch (1) und (2) ungeändert.

*Theorem IV.—Die trilineare Form*

$$\begin{aligned} nn'n'' - n \sum y'_{ik} y''_{ik} - \sum_1^4 x_i x'_i x''_i + \sum (x_i + x_k) y'_{ik} y''_{ik} - 2 \sum y_{ik} y'_{ik} y''_{ik} \\ - n' \sum y_{ik} y''_{ik} \qquad + \sum (x'_i + x'_k) y_{ik} y''_{ik} \\ - n'' \sum y_{ik} y'_{ik} \qquad + \sum (x''_i + x''_k) y_{ik} y'_{ik} \end{aligned} \tag{4}$$

*bleibt ungeändert, wenn auf jede der 3 Variabelnreihen die linearen Substitutionen* (1) *angewendet werden.*

Die Form (4) drückt die Anzahl der Schnittpunkte dreier $M_3^n$, $M_3^{n'}$, $M_3^{n''}$ aus, welche die bezüglichen Vielfachheiten in $a_i$, $y_{ik}$ besitzen. Um sie zu rechnen, beachte man, dass für die Schnittcurve von $M_3^{n'}$, $M_3^{n''}$ wird

$$\begin{aligned} \mathfrak{n} &= n'n'' - \sum y'_{ik} y''_{ik}, \\ \mathfrak{x}_i &= x'_i x''_i - y'_{ik} y''_{ik} - y'_{il} y''_{il} - y'_{im} y_{im}, \\ \mathfrak{y}_{ik} &= y'_{ik} (n'' - x''_i - x''_k + y''_{ik}) + y''_{ik} (n - x'_i - x'_1 + y'_{ik}) \end{aligned}$$

und setze in (3) statt $\mathfrak{n}$, $\mathfrak{x}_i$, $\mathfrak{y}_{ik}$ diese Werthe.

*Theorem V.—Die Substitutionen* (1) *lassen die Form*

$$n^3 - 3n \sum y_{ik}^2 - \sum x^3 + 3 \sum (x_i + x_k) y_{ik}^2 - 2 \sum y_{ik}^3 \tag{5}$$

*ungeändert.*

Diese Form entsteht, wenn die drei Variabelnreihen in (4) identisch gemacht werden, was bekanntlich eine Covariante der trilinearen Form gibt.

*Theorem VI.—Die Substitutionen* (1) *lassen das Werthesystem* $n = 4s$, $x_1 = x_2 = x_3 = x_4$ *ungeändert.*

Man überzeugt sich hievon durch Ausrechnung und schliesst:

*Corollar.* Lässt (1) den Singularitätencomplex $n$, $x_i$, $y_{ik}$ ungeändert, so lässt es auch den Singularitätencomplex $n - 4$, $x_i - 2$, $y_{ik} - 1$ ungeändert.

*Theorem VII.—Die Substitutionen* (1) *lassen die Form*

$$\begin{aligned} (n+1)(n+2)(n+3) - \sum x_i (x_i + 1)(x_i + 2) \\ + \sum y_{ik} (y_{ik} + 1)(3x_i + 3x_k - 4y_{ik} + 4) - \sum y_{ik} (y_{ik} + 1)(3n - 2y_{ik} + 5) - 6 \end{aligned} \tag{6}$$

*ungeändert.*

Dasselbe drückt das 6 fache der Dimension von $M_3^n$ aus,* also eine inva-

* Nach der Angabe von Nöther, Ann. di Mat. V, wenn man daselbst die $k_\nu = 0$ setzt. Herrn Nöther's Resultate selbst sind hier sonst nirgends verwendet.

riante Zahl. Es folgt hieraus sofort, was man aber eben auch durch directe Einsetzung verificirt.

*Theorem VIII.—Die Substitutionen* (1) *lassen die Formen*

$$2n^2 - \sum x_i^2, \tag{7}$$

$$4n - \sum x_i \tag{8}$$

*ungeändert.*

Ein anderer Beweis ist, dass man VI mit dem Singularitätencomplexe $\mathfrak{n}$, $\mathfrak{x}_i$, $\mathfrak{y}_{ik}$ was IV schneidet, wo $M_2^{\mathfrak{n}'} = M_2^{\mathfrak{n}''}$ gesetzt ist. Auch die Polarisirung von (5) und (7) nach VI liefert (7) und (8).

*Theorem IX.—Die Substitutionen* (1) *lassen auch die Formen*

$$11n - 2\sum x - \sum y_{ik}, \tag{9}$$

$$n^2 - \sum x^2 + \sum y_{ik}^2 \tag{10}$$

*ungeändert.*

Es folgt auch aus VII durch Zerlegung in die homogenen Bestandtheile.

*Corollar.* Man setzt überdies aus (10), (7), (9) die Form zusammen

$$(n+1)(n+2)(n+3) - \sum x_i(x_i+1)(x_i+2) + \sum y_{ik}(y_{ik}-1)(y_{ik}+1).$$

*Theorem X.—Die Substitutionen* (1) *lassen jede der* 6 *Formen* $x_i - x_k$, *sowie*

$$\sum y_{ik}^2(n - x_i - x_k + y_{ik});\ \sum y_{ik}(n - x_i - x_k + y_{ik})^2;\ \sum y_{ik}(y_{ik} + n - x_i - x_k)$$

*ungeändert.*

Es folgt aus (10) und (5).

*Theorem XI.—Die Substitutionen* (1) *lassen jede der Formen* $F(y_{ik};\ n - x_l - x_m + y_{lm})$ *ungeändert, wenn* $F$ *symmetrisch in Bezug auf seine beiden Argumente ist.*

Denn es wird $y_{ik}$ mit $n - x_l - x_m + y_{lm}$ vertauscht.

*Theorem XII.—Die Substitutionen* (2) *lassen die Formen*

$$\mathfrak{n}^2 - 2\sum \mathfrak{x}_i^2, \tag{11}$$

$$2\mathfrak{n} - \sum \mathfrak{x}_i \tag{12}$$

*und das Werthesystem* $\mathfrak{n} = 4s$, $\mathfrak{x}_i = s$ *oder* $\mathfrak{n} = 6s$, $\mathfrak{y}_{ik} = 2s$ *oder* $\mathfrak{x} = 10s$, $\mathfrak{x}_i = s$, $\mathfrak{y}_{ik} = 2s$ *ungeändert.*

Ebenso bleiben die symmetrischen Formen von $\mathfrak{y}_{ik}$ und $\mathfrak{y}_{lm}$ ungeändert und immer gleichzeitig $\mathfrak{n}$, $\mathfrak{x}_i$ und $\mathfrak{n}-4$, $\mathfrak{x}_i - 1$ oder $\mathfrak{n}-6$, $\mathfrak{y}_{ik}-2$ oder $\mathfrak{n}-10$, $\mathfrak{x}_i - 1$, $\mathfrak{y}_{ik} - 2$.*

Zum Schlusse sollen noch einige geometrische Eigenschaften der Transformation angegeben werden.

*Theorem XIII.—Die osculirenden Kegel von $M_2^3(b_1^2 \dots b_4^2)$ in $b_1 \dots b_4$ schneiden die Gegenflächen in Kegelschnitten, durch welche eine $M_2^2$ geht, deren Tangentenebenen in $b_1 \dots b_4$ die Gegenflächen in Geraden einer Ebene $M_2^1$ schneidet.*

Ist $\sum a_i x_k x_l x_m = 0$ die $M_2^3$, so ist $\sum a_i a_k x_l x_m = 0$ die $M_2^2$ und $\sum a_k a_l a_m x_i = 0$ die $M_2^1$.

*Corollar.* Die $M_2^2$ durch $a_1 a_2 a_3 a_4$, deren Tangentenebenen in $a_1$, $a_2$, $a_3$, $a_4$ die Gegenebenen in 4 Geraden einer $M_2^1$ schneiden, werden durch die Transformation in ebensolche $M_2^2$ durch $b_1$, $b_2$, $b_3$, $b_4$ verwandelt.

Die $M_2^2$ dieser Eigenschaft bilden innerhalb des $R_5$ der $M_2^2$ durch $a_1 a_2 a_3 a_4$ eine $M_3^4$, welche auf den $R_3$ abbildbar ist, sodass die linearen Schnitte der $M_3^4$ durch die Flächen 2. Classe abgebildet werden, welche die 4 Ebenen von $a_1 a_2 a_3 a_4$ berühren. Die Transformation $(a_i;\, b_i)^3$, welche im $R_5$ der $M_2^2$ eine Collineation bewirkt, ruft also eine invariante $M_3^4$ des $R_5$ hervor.—Auch besteht unter den dreifach berührenden Ebenen der $M_2^3$ und ihren entsprechenden Ebenen eine $(a_i;\, b_i)^3$, welche nach beiden Richtungen dieselbe ist.

*Theorem XIV.—Wenn in einer Transformation mit irgend einer Characteristik eine $M_2^2$ invariant bleibt, so liefert die stereographische Projection der $M_2^2$ eine ebene Transformation 5. Ordnung mit der Characteristik Ch. $(e_3,\, e_4,\, e_5,\, e_6;\, e_3',\, e_4',\, e_5',\, e_6')$ nebst $(e_1 e_1')(e_2 e_2')$ oder $(e_1 e_2')$, $(e_2 e_1')$, wo die Characteristik der $e_i, e_i'$ gleich räumlichen der Characteristik Ch $(a_1,\, a_2,\, a_3,\, a_4;\, b_1,\, b_2,\, b_3,\, b_4)$ ist.*

Eine Gerade der Ebene hat als Bild auf $M_2^2$ einen Kegelschnitt; dieser verwandelt sich in eine $C_6\,(b_1^2 b_2^2 b_3^2 b_4^2)$ durch $O$, wenn $O$† ein Doppelpunkt ist, und diese wird in $C_5$ projicirt, welche 4 Doppelpunkte und in den Fusspunkten der

---

* Ich erwähne noch die Theoreme: *Wenn eine $M_2^n$ in den Kanten $a_i a_k$ Vielfachheiten hat, welche eine Consequenz der $x_i$ sind, so haben die adjungirten $M_2^{n-4}$ in den $a_i a_k$ Vielfachheiten, welche gleich denen von $M_2^n$ sind.*

Jene Vielfachheiten sind $x_i + x_k - n$, diejenigen von $M_2^{n-4}$ sind $x_i - 2 + x_k - 2 - (n-4) = x_i + x_k - n$. *Eine $M_2^n$, welche in $a_i a_k$ nur die consequenten Vielfachheiten besitzt, verwandelt sich in eine $M_2^k$ mit $y_{ik} = 0$.*

† Für die Voraussetzung, dass $O$ kein Doppelpunkt ist, welche discutirt werden muss, wenn es sich nicht um Characteristiken, sondern um Transformationen handelt, sehe man §12.

zwei Erzeugenden durch $O$ zwei weitere Doppelpunkte hat. Hiebei wird jede Erzeugende in eine $M_1^2$ verwandelt, welche entweder sie selbst oder die andere Erzeugende zweimal trifft, welchem $(e_1 e_2')(e_2 e_1')$ oder $(e_1 e_1')(e_2 e_2')$ entsprechen.— Eine solche Transformation 5. O. wird übrigens schon erhalten, wenn man zwei entsprechende Flächen 2. O. aus zwei entsprechenden Punkten stereographisch projicirt.*

*Theorem XV.—Die Theorie der Characteristiken von* $(a_i;\ b_i)^3$ *ist auch in Hinsicht auf Aequivalenz identisch mit der Theorie der ebenen Characteristiken* 5. *O.*, *welche zwei Paare Fundamentalpunkte* $(e_1 e_1')(e_2 e_2')$ (oder $(e_1 e_2')(e_2 e_1')$) *gemeinsam in Coincidenz haben.*

Ich denke mir nämlich, dass durch die sämmtlichen Punkte der Characteristik eine Fläche 2. O. gehe, welche in sich transformirt wird, obzwar thatsächlich die Anzahl $\sigma$ der Punkte $> 9$ ist, und kann also die Projection der $M_2^2$ aus einem Punkte $O$ vornehmen unter Beachtung der Verträglichkeit mit der Natur von $(a_i;\ b_i)^3$. Da wir nur Transpositionen anwenden, welche ebenfalls $(a_i;\ b_i)^3$ sind, so kann gedacht werden, dass auch diese dieselbe $M_2^2$ invariant lassen und es entspricht ihnen also in der Ebene je eine Transposition der 5. O. Ebenso gilt:

*Theorem XVI.—Die Gruppen von* $(a_i;\ b_i)^3$ *können genau abgeleitet werden aus den Gruppen ebener Transformationen* 5. *O.*, *welche dieselben beiden Coincidenzen* $(e_1 e_2')$, $(e_2 e_1')$ *oder* $(e_1 e_1')$, $(e_2 e_2')$ *besitzen und gemeinsam haben.*†

### § 2.—*Zusammensetzung. Periodische Characteristiken. Aequivalenz.*

1. Das Problem der successiven Zusammensetzung führt nun sofort zu dem Probleme der periodischen Characteristiken. Um Characteristiken der Substitutionen (1) zu schreiben, hat man $n'$ dem $n$ entsprechend zu setzen, dann die $x'$ mit den $x$ und die $y'$ mit den $y$ zu verketten. Rein arithmetisch wäre es wohl denkbar, sogar die $x'$ mit den $y$ und die $y'$ mit den $x$ zu verketten, sofern nicht die Verschiedenheit der Coefficienten in den anallagmatischen Formen entgegen-

* Dies gilt jedoch nur, wenn $M_2^2$ durch keine Kante $a_i\, a_k$ geht. Wenn sie durch 1, 2, 3, 4 derselben geht, erniedrigt sich die Ordnung der Transformation in der Ebene um 1, 2, 3, 4. Dass es nicht erlaubt ist, für *alle* Characteristiken die Annahme zu machen, lehrt der Erfolg. Cf. auch die I. Note a. E. meines Buches (Berlin, Mayer & Müller, 1895): Theorie der endlichen Gruppen eindeutiger Transformationen in der Ebene.

† Es ist aber zu bemerken, dass man in dieser gewissermassen imaginären Ableitung die Fläche 2. O. auch durch irgend eine abbildbare Fläche ersetzen kann, welche einen durch die Characteristik invarianten Singularitätencomplex besitzt.

steht. Damit die Characteristik periodisch sei, ist nöthig, dass die $b_i$ mit den $a_i$ coincidiren oder verkettet sind. Also können die Fälle, wo $b_i$ mit Kanten $a_i a_k$ incident sind, ohne dass sie hiedurch in Fundamentalpunkte des anderen Raumes übergehen, nicht periodisch sein. Wir werden also nur die Coincidenz und Verkettung der Fundamentalpunkte zu untersuchen haben. Die Aequivalenz wird mittelst $(a_i; b_i)^3$ untersucht werden, welche ihre Fundamentalpunkte über der Characteristik oder gewöhnlichen Punkten haben und gipfelt in dem Probleme: Die Typen anzugeben, welchen alle periodischen Characteristiken durch $(a_i; b_i)^3$ äquivalent sind.

Zu jeder Characteristik gehören characterisirte lin. Substitutionen (1) und (2). Dieselbe ist periodisch, wenn die characteristische Function einfache Elementartheiler und nur Einheitswurzeln zu Wurzeln hat (Theorem von Frobenius).*

Ich anticipire hier ein Theorem, dessen Beweis mit Hilfe der einzelnen Typen, welche im §7 aufgestellt werden, geführt werden kann:

*Theorem XVII.—Wenn eine Characteristik von* $(a_i; b_i)^3$ *periodisch ist, so sind alle Potenzen derselben, welche gleichen Index haben, mit ihr durch Transpositionen* $(a_i; b_i)^3$ *äquivalent.*

2. Um für die vollständigen Substitutionen (1) die Characteristik zu bilden, hat man ausser für die Verkettung der $x'$ mit den $x$ auch für die Verkettung der $y'_{ik}$ mit den $y_{ik}$ zu sorgen. Arithmetisch ist es zulässig, über diese letztere Verkettung ganz unabhängig von der ersteren zu verfügen, und dadurch entsteht eine ganz neue und sicherlich eigenartige Reihe periodischer linearer Substitutionen. Für jene jedoch, welche geometrischen Transformationen entsprechen, gilt:

*Theorem XVIII.—Durch die Coincidenzen und Verkettungen der Fundamentalpunkte sind die Verkettungen der Fundamentalgeraden* $a_i a_k$ *vollständig bestimmt.*

Dies folgt daraus, dass die Transformation durch die Fundamentalpunkte und ein Paar entsprechender Punkte bestimmt ist, und gleichzeitig:

*Theorem XIX.—Die Characteristik ist schon periodisch, wenn auch nur die unvollständigen Substitutionen* (1) *periodisch sind. Die vollständigen Substitutionen* (2) *sind periodisch, wenn auch nur die unvollständigen es sind.*

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*Cr. J., Bd. 84: "Ueber bilineare Formen und lineare Substitutionen."

Da die vollständigen (1) eine Determinante haben, die in das Product der Determinante der unvollständigen (1) und Factoren $(x^i - 1)$ zerfällt, so folgt XVIII auch aus dieser characteristischen Function. Ebenso gilt, dass die characteristische Function der vollständigen (2) sich aus der Determinante der unvollständigen und aus Factoren $x^i - 1$ zusammensetzt.

*Theorem XX.—Die successiven Transformationen irgend welcher Characteristik haben ungerade Ordnungszahlen und gerade Vielfachheitszahlen.*

Denn die Ordnung kann sich nur um Vielfache gerader Zahlen vermindern und durch Multiplication mit ungeraden Zahlen vergrössern. Die Vielfachheiten vergrössern sich durch Multiplication mit 2 und vermindern sich um Vielfache gerader Zahlen. Dasselbe gilt für die *homaloidalen Curven* in den successiven Transformationen.

3. Wie für die quadratischen Transformationen im $R_r$† beweise ich auch hier: Die Substitutionen (1) und (2) sind stets gleichzeitig periodisch oder nicht. Dies wird in §7 präcisirt werden.

4. Die Theoreme aus Cr. J., Bd. CXIV, über die uneigentlichen Cyclen, über die Invarianten der Substitutionen, ihre geometrische Bedeutung und über die Bedingungen der Aequivalenz gelten auch hier.

## §3.—*Die Characteristiken ohne Coincidenzen oder mit* $(a_1 b_1)$.

1. *Theorem XXI.—Die Characteristik* $b_1$ *in* $a_1$, $b_2$ *in* $a_2$, $b_3$ *in* $a_3$, $b_4$ *in* $a_4$ *ist aperiodisch.*

1. Beweis. Die successiven Fundamentalsysteme sind: Ebene, $(2, 2, 2, ., 2, . 2 .)^8$, $(6, 2, 6, 2, 6, 2, 6, 2)^9$, $(12, 6, 12, 6, 12, 6, 12, 6)^{19}$, $( . )^{33}$, $( . )^{51}$, .... Die Ordnungsreihe hat als allgemeines Glied $2s^2 + 1$.

2. Beweis. Die characteristische Function der Substitution (1) hat den Werth $\rho^8 (\rho - 3)$.

*Theorem XXII.—Alle Characteristiken* $b_i$ *in* $a_k$, *wo die* $k$ *eine beliebige Permutation der* $i$ *bilden, sind aperiodisch.*

Denn die Ordnungen und Vielfachheiten bleiben dieselben wie für $k = i$.

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* Ich werde also die Characteristiken immer nur durch die Fundamentalpunkte bezeichnen.

† Rendiconti Ist. Lomb. 31. Mai 1894, "Sulle caratteristiche delle trasformazioni quadratiche nello spazio a $r$ dimensioni."

*Theorem XXIII.—Alle Characteristiken ohne Coincidenzen sind aperiodisch.*
Denn das Theorem für die ebenen Transformationen, dass, wenn man in einer aperiodischen Characteristik die Verkettungen erweitert, die Characteristik aperiodisch bleibt, gilt auch hier.

2. Man könnte vermuthen, dass in XXI die nöthigen Reductionen in der Ordnung dadurch ermöglicht werden könnten, dass Kanten $b_i b_k$ durch Punkte $a$ gehen. Eine leichte Rechnung zeigt jedoch, dass es weder hinreicht, ausser der Verkettung in XXI noch Incidenz von $b_1 b_2$, $b_2 b_3$, $b_3 b_4$, $b_4 b_1$ mit $a_1$, $a_2$, $a_3$, $a_4$ anzunehmen, noch etwa $(a_1 a_1')$ nebst Incidenz von $b_2 b_3$, $b_3 b_4$, $b_4 b_2$ mit $a_2$, $a_3$, $a_4$. A fortiori ist zu schliessen:

*Theorem XXIV.—Keine aperiodische Characteristik kann dadurch periodisch werden, dass man Incidenzen von $b_i b_k$ mit Punkten $a_l$ nachträglich einträgt.*

Auch die Substitutionen (2) stützen diese Behauptung. Denn wenn auch die Incidenzen die Ordnungen der Flächen erniedrigen, so tragen sie nicht zur Verminderung der successiven Ordnungen der Curven bei, da diese die Kanten $b_i b_k$ in variabeln Punkten treffen, welche von dem incidenten Punkte verschieden sind.

3. Es ist gewiss, dass die Characteristiken mit $(a_1 b_1)$ nicht periodisch werden können, wenn nicht die Characteristik der quadratischen Transformation, welche dieselben Verkettungen von $b_2 b_3 b_4$ mit $a_2 a_3 a_4$ besitzt, periodisch ist. Es gilt aber auch:

*Theorem XXV.—Die Characteristik mit $(a_1 b_1)$ ist periodisch mit demselben Index, sobald die quadratische Characteristik, welche aus dem übrigen Theile zusammengesetzt ist, periodisch ist.*

Durch einige Zerlegungen verwandelt man die char. Function von (1) für $(a_1 b_1)$ in das Product aus $(\rho - 1)$ und der char. Function von $(a_2 a_3 a_4,\ b_2 b_3 b_4)^2$.

*Corollar.* Der Index der cubischen Characteristik ist gleich dem Index der quadratischen Characteristik Ch. $(a_2 a_3 a_4;\ b_2 b_3 b_4)$, welche ein Bestandtheil jener ist.

*Theorem XXVI.—Sind die Ordnungen der successiven Transformationen von Ch. $[a_2 a_3 a_4;\ b_2 b_3 b_4]$ $n_i$, so sind die successiven Ordnungen für $(a_1 b_1)$ Ch. $[a_2 a_3 a_4;\ b_2 b_3 b_4]$ $2n_i - 1$ mit $(a_1 b_1)^{2n_i - 2}$ und die übrigen Vielfachheiten sind das Doppelte der Vielfachheiten für die ebene Characteristik.*

Die Transformation von $M_2^{2n_i - 1}$ durch $T$ gibt die Ordnung $3(2n_i - 1) - (2n_i - 2)$

$-2\sum$, wenn $\Sigma$ die für die ebene Characteristik geltende Verminderung ist. Dann ist aber $n_{i+1}=2n_i-\sum$, $2n_{i+1}=4n_i-2\sum$, also die räumliche Ordnung $2n_{i+1}-1$. Nun gilt die Form für die Zusammensetzung zweier $T$, also stets.

*Corollar.* Die Absonderungsfläche, welche zu $(a_1 b_1)$ gehört, hat die Ordnung $n_i-1$ und die der übrigen Fundamentalpunkte Ordnungen gleich der Hälfte der Vielfachheiten.

Es gibt also 31 isolirte periodische Characteristiken und 9 Classen. Sie entstehen durch Verbindung der Characteristiken auf p. 83, Cr. J., Bd. CXIV mit $(a_1 b_1)$.

*Theorem XXVII.—Zwei Characteristiken $T$, $T_1$ mit $(a_1 b_1)$ sind äquivalent, wenn die quadratischen Characteristiken $R$, $R_1$, welche die Reste bilden, äquivalent sind.*

Denn wird $T$ mit $(a_1 b_1)$ $Q$ transponirt, wo $Q$ die $R$, $R_1$ äquivalent macht, so erhält man eine Transformation, welche unter den Geraden von $(a_1 b_1)$ eine quadratische Transformation hervorruft mit $R_1$ als Characteristik, welche also cubisch und $T_1$ ist.

*Corollar I.* Wenn $(R)$ der Collineation äquivalent ist, so ist auch $((a_1 b_1), R)^3$ der Collineation äquivalent.

*Corollar II.* Die Characteristiken $(a_1 b_1)$, $R$ sind reductibel in der Punktezahl, wenn $R$ in der Punktezahl reductibel ist.

*Corollar III.* Es gibt ebenso viele typische Characteristiken mit $(a_1 b_1)$, als es typische quadratische Characteristiken in der Ebene gibt.

*Theorem XXVIII.—Die typischen Characteristiken mit $(a_1 b_1)$ sind:* $(a_1 b_1)(B_{12})$; $(a_1 b_1)\,B_6$; $(a_1 b_1)\,B_9$; $(a_1 b_1)\,B_{12}$; $(a_1 b_1)\,B'_{12}$; $(a_1 b_1)\,B_{18}$; $(a_1 b_1)\,B_{15}$; $(a_1 b_1)\,B_{20}$; $(a_1 b_1)\,B_{24}$; $(a_1 b_1)\,B_{30}$; $(a_1 b_1)(a_2 b_2)\,b_3$ *in* $\ldots\, a_4$, $b_4$ *in* $\ldots\, a_3$.*

## §4.—*Die Characteristiken mit $(a_1 b_2)(a_2 b_1)$ und ihre Derivirten.*

*Theorem XXIX.—Die Characteristiken $(a_1 b_2)$, $(a_2 b_1)$, $b_3$ in $\ldots\, b_3^m = a_3$, $b_4$ in $\ldots\, b_4^n = a_4$ sind äquivalent der Collineation. Der Index ist das kleinste Multiplum von 2 und $m+n+2$.*

Denn $(a_1 b_1)$, $(a_2 b_2)$, $b_3$ in $\ldots\, a_3$, $b_4$ in $\ldots\, a_4$ war es wegen Ch. $(a_2 a_3 a_4\,;\, b_2 b_3 b_4)$ und aber durch eine Transposition, welche $a_1$, $a_2$ symmetrisch enthält, sodass die

* Unter den internen Characteristiken seien insbesondere die involutorischen $(14, 6, 6, 6, 6, 6, 6, 6)^{15}$ und $(32, 12, 12, 12, 12, 12, 12, 12, 12)^{33}$ hervorgehoben.

Vertauschung von $b_1$, $b_2$ die Transposition nicht beeinträchtigt. Der Index folgt aus dem von $(a_1 b_1)$, $(a_2 b_2)$.

*Theorem XXX.—Die Characteristiken* $(a_1 b_2)$, $(a_2 b_1)$, $b_3$ *in* .... $b_3^{m_1} = a_4$, $b_4$ *in* .... $b_4^{m_2} = a_3$ *sind irreductibel in Ordnung und Punktezahl und vom Index* $2N$, *wenn* $N$ *das kleinste Vielfache von* $m_1 + 1$ *und* $m_2 + 1$.

Denn eine Transposition mit $a_1$, $a_2$ würde auch $(a_1 b_1)$, $(a_2 b_2)$ reduciren, welche aber irreductibel ist. Die günstigste Transposition mit $a_1$ allein wäre $a_1 b_3 b_3' a_4$, welche nicht reducirt. Der Index folgt aus dem von $(a_1 b_1)$, $(a_2 b_2)$. Da char. Function ist $(\rho - 1)^2(\rho^{m_1+1} + 1)(\rho^{m_2+1} + 1)$.

*Theorem XXXI.—Die Characteristiken* $(a_1 b_2)$, $b_1$ *in* $a_2$, $b_3$ *in* $a_3$, .... *sind reductibel mit Characteristiken mit* $(a_1 b_1)$.

Die Transposition $(a_1 b_1 a_2 b_3)^3$ liefert* $A_3$ in $a_3$ in $b_1 b_2 b_4$ in $B_1 B_2 B_4$, $A_1$ in $b_1 a_2 b_3$ in $b_2^2 b_1 b_3 b_4 a_2 a_3$ in $B_2 B_4 A_3$, $A_2$ in $a_1 b_1 b_3$ in $b_1^2 a_2 a_3 b_2 b_3 b_4$ in $B_1 A_3 B_4$, $B_1$ in $a_1 a_2 b_3$ in $b_1 b_2 a_3$ in $B_1 B_2 A_3$, also $(A_3 A_3,\ A_1 B_1,\ A_2 B_2,\ B_1 B_4)^3$ mit $B_4$ in $B_4'$ in .... $A_4$ in $A_2$.

*Theorem XXXII.—Die Characteristik* $(a_1 b_2)$, $b_1$ *in* $a_2$, $b_3$ *in* $b_3'$ *in* $a_3$, $b_4$ *in* $b_4'$ *in* $a_4$ *ist aperiodisch.*

Die successiven Ordnungen sind $s^2 - s + 1 - \sum(6p)$, wo die Summe zu erstrecken ist über alle durch 6 theilbaren ganzen Zahlen $< 2(s-1)$.

*Theorem XXXIII.—Die Characteristiken* $(a_1 b_2)$, $b_1$ *in* $a_2$, $b_4$ *in* $a_3$, $b_3$ *in* .... $a_4$ *und* $(a_1 b_2)$, $b_1$ *in* $a_2$, $(a_3 b_4)$, $b_3$ *in* .... $a_4$ *sind äquivalent je einer Characteristik des* §6.

Die Transposition $(a_1 b_1 a_2 a_3)^3$ liefert $A_3$ in $a_1 b_1 a_2$ in $b_1 b_2 a_2$ in $A_3$, $A_1$ in $b_1 a_2 a_3$ in $b_2 b_3 a_2$ in $A_2 B_2 B_3$; $A_2$ in $a_1 b_1 a_3$ in $b_1 b_3 a_2$ in $B_1 A_2 B_3$; $A_4$ in $a_4$ in $b_1 b_2 b_3$ in $B_1 B_2 B_3$; $B_1$ in $a_1 a_2 a_3$ in $b_4$ in $B_1 A_1 A_2$, also $(A_1 B_1,\ B_1 B_3,\ A_2 A_1,\ A_4 A_2)^3$ mit $A_3$ in $A_3$ und $B_3$ in .... $A_4$ und ebenso für die andere Characteristik.

*Theorem XXXIV.—Die Characteristik* $(a_1 b_2)$, $b_1$ *in* $a_2$, $b_3$ *in* $b_3'$ *in* $a_4$, $b_4$ *in* $b_4'$ *in* $a_3$ *ist aperiodisch.*

Es folgt aus XXXI, weil die dortige Characteristik bezüglich $a_3$, $a_4$ symmetrisch ist.

*Theorem XXXV.—Die Characteristik* $(a_1 b_3)$, $b_1$ *in* $b_1'$ *in* $a_2$, $(a_3 b_4)$, $b_3$ *in* $b_3'$ *in* $a_4$ *ist aperiodisch.*

Die successiven Transformationen haben eine Ordnungsreihe, deren allge-

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* **Mit den Majuskeln werden die den Minuskeln entsprechenden Punkte des zweiten Raumes bezeichnet, in welchen transponirt wird.**

meines Glied ist $s^2 + 2s + 2$ oder $s^2 + 2s + 3$, je nachdem $s$ ungerade oder gerade.

*Theorem XXXVI.—Die Characteristik* $(a_1 b_2)$, $b_1$ *in* $b_1'$ *in* $a_2$, $b_2$ *in* $a_4$, $b_4$ *in* $a_3$ *ist aperiodisch.*

Die Reihe der Vielfachheiten von $b_3$ hat die Differenzenreihe $4s$, $4s + 2$, $4s + 2$, wo $s$ die natürliche Zahlenreihe in positiver Richtung durchläuft.

*Corollar.* Alle Characteristiken, welche aus $(a_1 b_2)$, $(a_2 b_1)$, $(a_3 b_i)$, $(a_4 b_k)$ abgeleitet wird, sind entweder äquivalent mit Characteristiken des §3 oder 6 oder sind aperiodisch.

### §5.—*Die Characteristiken mit* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_1)$ *und ihre Derivirten.*

*Theorem XXXVII.—Die Characteristiken* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_1)$, $b_4$ *in* .... $b_4^m = a_4$, *sind äquivalent der Collineation vom Index N, N das kleinste Multiplum von* 3 *und* $m + 1$.

Die Transposition $(a_1^2 a_2^2 a_3^2 d^2)^3$, wo $d$ ein gewöhnlicher Doppelpunkt ist, den man supponiren kann, liefern sofort die Collineation $A_1$ in $A_2$ in $A_3$ in $A_1$, $B_4$ in .... $A_4$.

*Theorem XXXVIII.—Die Characteristiken* $(a_1 b_3)$, $(a_2 b_3)$, $b_1$ *in* $a_3$; $b_4$ *in* .... $b_4^m = a_4$ *sind periodisch vom Index* $m + 5$ *und äquivalent der Collineation.*

Die Transposition $(a_1^2 a_2^2 a_3^2 b_1^2)^3$ liefert $B_1$ in $B_4$ in .... $A_4$ in $A_3$ in $A_2$ in $A_1$ in $B_1$.

*Theorem XXXIX.—Die Characteristiken* $(a_1 b_3)$, $(a_2 b_3)$, $b_1$ *in* $b_1'$ *in* $a_3$, $b_4$ *in* ... $b_4^m = a_4$, *sind periodisch für* $m = 0, 1, 2, 3, 4, 5$ *und äquivalent Characteristiken mit* $(a_1 b_1)$, $(a_2 b_3)$, $(a_3 b_4)$.

Die Transposition $(a_1^2 a_2^2 b_1^2 b_1'^2)^3$ liefert $A_3$ in $a_3$ in $b_1 b_2 b_4$ in $B_1 B_3 B_4$; $B_1$ in $a_1 a_2 b_1$ in $b_1 b_3 a_3$ in $B_1 B_3 A_3$; $B_4'$ in $a_1 a_2 b_1$ in $b_1 b_3 b_1'$ in $A_2$; $A_4$ in $a_4$ in $b_1 b_3 b_3$ in $B_1'$; $A_1$ in $a_2 b_1 b_1'$ in $b_3^2 b_1 b_3 b_4 b_1' a_3$ in $A_3 B_2 B_4$; $A_2$ in $a_3 b_1 b_4'$ in $b_1^2 b_2 b_3 b_4 b_1' a_3$ in $B_1 B_4 A_3$, also $(A_3 A_3, B_1 B_4, B_2 B_1, A_2 A_2)^3$ mit $B_4$ in .... $A_4$ in $B_1'$ in $A_2$.

*Theorem XL.—Die Characteristiken* $(a_1 b_2)$, $(a_2 b_3)$, $b_1$ *in* $b_1'$ *in* $b_1''$ *in* $a_3$, $b_4$ *in* .... $b_4^m = a_4$ *sind äquivalent Characteristiken* $(a_1 b_3)$, $(a_2 b_3)$, $b_4$ *in* $a_4$, ....

Die Transposition $(a_1^2 a_2^2 b_1^2 b_1'^2)^3$ liefert $(A_3 B_1'', B_1 B_4, B_3 B_1, A_2 B_2)^3$ mit $B_1''$ in $A_3$, $B_4$ in .... $A_4$ in $B_1'$ in $A_2$. Auch $(a_1^2 a_2^2 b_1^2 a_3^2)^3$ liefert das Theorem.

*Theorem XLI.—Die Characteristiken* $(a_1 b_2)$, $(a_2 b_3)$, $b_1$ *in* .... $a_3$, $b_4$ *in* $a_4$ *werden durch* $(a_1^2 a_2^2 b_4^2 a_4^2)^3$ *in andere gleicher Natur transformirt.*

Die Transposition liefert $A_1$ in $a_2 b_4 a_4$ in $b_2 b_4 a_4$ *in* $A_2$, $A_2$ in $a_1 b_4 a_4$ in $b_1 b_4 a_1$ in $B_1 B_4 A_4$, $B_4$ in $a_1 a_2 a_4$ in $b_3$ in $A_1 B_4 A_4$, $A_4$ in $a_1 a_2 b_4$ in $b_1 b_2 a_4$ in $B_1 A_1 A_4$, $A_3$ in $a_3$ in $b_1 b_2 a_4$ in $B_1 A_1 B_4$, also $(A_2 A_1, B_4 B_1, A_4 B_4, A_3 A_4)^3$ mit $A_1$ in $A_2$, $B_1$ in .... $A_3$.

*Theorem XLII.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $b_1$ *in* $b_1'$ *in* $b_1''$ *in* $a_3$, $b_4$ *in* $a_4$ *ist aperiodisch.*

Die Ordnungsreihe hat als allgemeines Glied $\frac{1}{2}(3s^2 + 6s + 5)$, $\frac{1}{2}(3s^2 + 10s + 13)$, $\frac{1}{2}(3s^2 + 14s + 21)$.

*Theorem XLIII.—Die Characteristiken* $(a_1 b_2)$, $b_3$ *in* $a_2$, $b_1$ *in* $a_3$, .... *sind äquivalent Characteristiken des* §3.

Die Transposition $(a_1^2 b_3^2 a_2^2 b_1^2)^3$ liefert $A_1$ in $a_1$ in $b_3 a_2 b_1$ in $b_2^2 b_1 b_3 b_4 a_3 a_2$ in $B_2 B_4 A_3$; $B_3$ in $a_4 a_2 a_1$ in $b_1 b_2 a_3$ in $A_1 B_1 A_3$; $A_2$ in $a_1 b_3 b_1$ in $b_1^2 b_2 b_3 b_4 a_2 a_3$ in $B_1 A_3 B_4$; $B_1$ in $a_1 b_3 a_2$ in $b_1 b_2 a_2$ in $A_1$; $A_3$ in $a_3$ in $b_1 b_2 a_4$ in $A_1 B_1 B_4$, also $(A_1 B_1, B_3 B_4, A_2 A_1, A_3 A_2)^3$ mit $B_1$ in $B_3$, $B_4$ in .... $A_4$ in $A_2$.

*Theorem XLIV.—Die Characteristik* $(a_1 b_2)$, $b_3$ *in* $b_3'$ *in* $a_2$, $b_1$ *in* $b_1'$ *in* $a_3$, $b_4$ *in* $a_4$ *ist aperiodisch.*

Die Ordnungsreihe hat die Differenzenreihe 2, 4, 8, 14, 22, 38, 62, 102, 168, 274, ...., welche ersichtlich wachsend ist.

*Conclusion.* Alle Characteristiken, welche Derivirte von $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_1)$, $(a_4 b_4)$ sind, sind entweder aperiodisch oder äquivalent mit Characteristiken $(a_1 b_1)$.

## §6.—*Die Derivirten von* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $(a_4 b_1)$.

*Theorem XLV.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $(a_4 b_1)$ *ist nicht äquivalent einer Collineation.*

Es gibt kein homaloidales System, das durch sie in sich selbst verwandelt werden könnte, denn dasselbe müsste in $a_1$, $a_2$, $a_3$, $a_4$ dieselbe Vielfachheit $x$ besitzen, daher, wenn es von der Ordnung $n$ ist, $3n - 4x = n$ oder $n = 2x$. Nun werden aber, wie hier nicht ausgeführt werden soll, die Ordnungen, welche hier zur Verfügung stehen (cf. meine Arbeit in Acta Math., 1897), ungerade, also nicht anwendbar.

*Theorem XLVI.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ *in* $a_4$ *ist periodisch mit dem Index* 6 *und typisch.*

Die successiven Transformationen sind $(2, 2, 2, 2, .)^3$, $(2, 2, ., 2, 2)^3$, $(2, ., 2, 2, 2)^3$, $(., 2, 2, 2, 2)^3$, $(2, 2, 2, ., 2)^3$, $(.)^1$. Die Irreductibilität folgt aus demselben Grunde wie vorher, ebenso bei den übrigen in diesem § zu erwähnenden Typen.

*Theorem XLVII.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $b_4$ *in* $a_3$, $b_1$ *in* $a_4$ *ist periodisch mit dem Index* 8 *und typisch.*

Die successiven Transformationen sind (2, 2, 2, . , 2, . )$^{3}$, (4, 2, 2, 2, 4, 2)$^{5}$, (2, 2, 2, 2, 4, 4)$^{5}$, (2, 2, 4, 2, 2, 4)$^{5}$, (2, 2, 4, 4, 2, 2)$^{5}$, (2, 4, 2, 4, 2, 2)$^{5}$, (2, 2, . , 2, . , 2)$^{3}$, ( . )$^{1}$.

*Theorem XLVIII.—Die Characteristik* $(a_1 b_2)$, $b_3$ *in* $a_2$, $b_4$ *in* $a_3$, $b_1$ *in* $a_4$ *ist periodisch vom Index* 30.

Die successiven Transformationen sind (2, 2, . , 2, . , 2, . )$^{3}$, (4, 4, 2, 4, 2, 6, 2)$^{7}$, (6, 6, 4, 6, 4, 8, 6)$^{11}$, (6, 6, 6, 8, 6, 8, 8)$^{13}$, (6, 6, 6, 8, 8, 6, 8)$^{13}$, (4, 6, 6, 6, 8, 4, 6)$^{11}$, (4, 6, 6, 4, 6, 2, 2, 4)$^{9}$, (4, 4, 6, 2, 4, 2, 2)$^{7}$, (4, 2, 4, . , 2, 2, 2)$^{5}$, (2, . , 2, . , . , 2, 2)$^{3}$, (2, . , . , 2, . , 2, 2)$^{3}$, (2, 2, . , 4, 2, 4, 2)$^{5}$, (4, 6, 2, 6, 4, 5, 4)$^{9}$, (6, 8, 6, 8, 6, 8, 6)$^{13}$, (8, 8, 8, 8, 8, 8, 8)$^{15}$, (6, 6, 8, 6, 8, 6, 8)$^{13}$, (4, 4, 6, 4, 6, 2, 6)$^{9}$, (2, 2, 4, 2, 4, . , 2)$^{5}$, (2, 2, 2, . , 2, . , . )$^{3}$, (2, 2, 2, . , . , 2, . )$^{3}$, (4, 2, 2, 2, . , 4, 2)$^{5}$, (4, 2, 2, 4, 2, 6, 4)$^{7}$, (4, 4, 2, 6, 4, 6, 6)$^{9}$, (4, 6, 4, 8, 6, 6, 6)$^{11}$, (6, 8, 6, 8, 8, 6, 6)$^{13}$, (6, 8, 8, 6, 8, 6, 6)$^{13}$, (6, 6, 8, 6, 6, 4, 4)$^{11}$, (4, 2, 6, 2, 4, 2, 4)$^{7}$, (2, . , 2, . , 2, . , 2)$^{3}$, ( . )$^{1}$.

*Theorem XLIX.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $b_4$ *in* $a_3$, $b_1$ *in* $b'_1$ *in* $a_4$ *ist periodisch vom Index* 14 *und typisch.*

Die successiven Transformationen sind (2, 2, 2, . , 2, . , . )$^{3}$, (4, 2, 2, 3, 4, 2, . )$^{5}$, (4, 4, 2, 2, 6, 4, 2)$^{7}$, (6, 4, 4, 2, 6, 6, 4)$^{9}$, (6, 4, 6, 4, 8, 6, 6)$^{11}$, (6, 6, 8, 6, 8, 8, 6)$^{13}$, (8, 8, 8, 8, 8, 8, 8)$^{15}$, (6, 6, 6, 8, 6, 8, 8)$^{13}$, (4, 6, 6, 6, 4, 6, 8)$^{11}$, (4, 4, 6, 6, 2, 4, 6)$^{9}$, (2, 4, 4, 6, 2, 2, 4)$^{7}$, (2, 4, 2, 4, . , 2, 2)$^{5}$, (2, 2, . , 2, . , . , 2)$^{3}$, ( . )$^{1}$.

*Theorem L.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $b_4$ *in* $b'_4$ *in* $a_3$, $b_1$ *in* $b'_1$ *in* $a_4$ *ist aperiodisch.*

Das allgemeine Glied der Ordnungsreihe ist $2s^2 + 2s + 1 - \sum(8p)$ und $2s^2 + 4s + 1 - \sum(8p)$, je nachdem $s$ gerade oder ungerade, und wo die Summe sich auf alle durch 8 theilbaren Zahlen $< 2s$ bezieht.

*Theorem LI.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $b_4$ *in* $a_3$, $b_1$ *in* $b'_1$ *in* $b''_1$ *in* $a_4$ *ist aperiodisch.*

Die 1. Differenzenreihe der Ordnungen ist 2, 2, 4, 4, 6, 8, 8, 8, 10, 12, 12, 14, 14, 16, .... Einen anderen Beweis liefert die im Theorem XIV enthaltene Methode, indem die durch Projection erhaltene ebene Characteristik aperiodisch wird. Einen 3. Beweis liefert der Umstand, dass die interne Involution von XLIX (8, .... 8)$^{15}$ ist.

*Theorem LII.—Die Characteristik* $(a_1 b_3)$, $b_3$ *in* $a_2$, $b_4$ *in* $a_3$, $b_1$ *in* $b'_1$ *in* $a_4$ *ist aperiodisch.*

Die Projection einer $M_3^2$ durch die 8 Punkte liefert eine mit $b_1$ in $a$, $b$ in $a_1$, $b_2$ in $b'_2$ in $a_2$, $(a_3 b_3)$, $(a_4 b_4)$ äquivalente Characteristik, welche also aperiodisch ist.

*Theorem LIII.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ *in* $b'_1$ *in* $a_4$ *ist periodisch vom Index* 10 *und typisch.*

Die successiven Transformationen sind $(2, 2, 2, 2, . . , .)^3$, $(2, 2, . , 2, 2, .)^3$, $(4, 2, 2, 4, 2, 2)^5$, $(2, 2, 2, 4, 4, 2)^5$, $(4, 4, 4, 4, 4, 4)^7$, $(2, 2, 2, 2, 4, 4)^5$, $(2, 2, 4, 2, 2, 4)^5$, $( . , 2, 2, . , 2, 2)^3$, $(2, 2, 2, . , . , 2)^3$, $( . )^1$.

*Theorem LIV.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ *in* $b'_1$ *in* $b''_1$ *in* $a_4$ *ist periodisch vom Index* 18.

Die successiven Transformationen sind $(2, 2, 2, 2, . , . , . , .)^3$, $(2, 2, . , 2, 2, . , .)^3$, $(4, 2, 2, 4, 2, 2, .)^5$, $(4, 4, 2, 6, 4, 2, 2)^7$, $(6, 4, 4, 6, 6, 4, 2)^9$, $(6, 6, 4, 8, 6, 6, 4)^{11}$, $(8, 6, 6, 8, 8, 6, 6)^{13}$, $(6, 6, 6, 8, 8, 8, 6)^{13}$, $(8, 8, 8, 8, 8, 8, 8)^{15}$, $(6, 6, 6, 6, 8, 8, 8)^{13}$, $(6, 6, 8, 6, 6, 8, 8)^{13}$, $(4, 6, 6, 4, 6, 6, 8)^{11}$, $(4, 4, 6, 2, 4, 6, 6, 8)^9$, $(2, 4, 4, 2, 2, 4, 6)^7$, $(2, 2, 4, . , 2, 2, 4)^5$, $( . , 2, 2, . , . , 2, 2)^3$, $(2, 2, 2, . , . , . , 2)^3$, $( . )^1$.

*Theorem LV.—Die Characteristik* $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ *in* $b'_1$ *in* $b''_1$ *in* $b'''_1$ *in* $a_4$ *ist aperiodisch.*

Sei es die Projection einer $M_3^2$, wodurch der Typus $(a b)$, $(a b)$, $(a_3 b_3)$, $(a_4 b_4)$, $b_2$ in $b'_2$ in $b''_2$ in $b'''_2$ in $a_2$ der Ebene erhalten wird, sei es die interne Involution von LIV beweisen das Theorem wie LI.

*Conclusion.* Alle Characteristiken mit mehr als 7 Punkten, welche durch Verkettung aus $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $(a_4 b_1)$ abgeleitet sind, sind aperiodisch.

## §7.—*Uebersicht der periodischen Typen.*

*Theorem LVI.—Alle periodischen Characteristiken der cubischen Reciprokaltransformation lassen sich mittelst eben solcher Transpositionen äquivalent machen folgenden Typen oder Classen:*

1. $(a_1 b_1)$, $(a_2 b_2)$, $b_3$ in . . . . $b_3^{k_1} = a_4$, $b_4$ in . . . . $b_4^{k_2} = a_3$.
2. $(a_1 b_2)$, $(a_2 b_1)$, $b_3$ in . . . . $b_3^{k_1} = a_4$, $b_4$ in . . . . $b_4^{k_2} = a_3$.

3. $(a_1 b_1) B_6$ 4. $(a_1 b_1)(B_{12})$ 5. $(a_1 b_1) B_9$ 6. $(a_1 b_1) B_{12}$ 7. $(a_1 b_1) B'_{12}$ 8. $(a_1 b_1) B_{14}$ 9. $(a_1 b_1) B_{18}$ 10. $(a_1 b_1) B_{20}$ 11. $(a_1 b_1) B_{24}$ 12. $(a_1 b_1) B_{15}$ 13. $(a_1 b_1) B_{30}$ 14. $(a_1 b_1)(B_{18})$ 15. $(a_1 b_1)(B_{30})$.*

* In n. 8–15 sind also die Characteristiken $B$ zu bilden mit den Punkten $a_2$, $a_3$, $a_4$; $b_2$, $b_3$, $b_4$, wie es die Theorie der ebenen quadratischen Characteristiken (Cr. J., Bd. CXIV, p. 87) vorschreibt.

16. $(a_1 b_2)(a_2 b_3)(a_3 b_4)(a_4 b_1)$.
17. $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ in $a_4$.
18. $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ in $b_1'$ in $a_4$.
19. $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ in $b_1'$ in $b_1''$ in $a_4$.
20. $(a_1 b_2)$, $(a_2 b_3)$, $b_4$ in $a_3$, $b_1$ in $a_4$.
21. $(a_1 b_2)$, $(a_2 b_3)$, $b_4$ in $a_3$, $b_1$ in $b_1'$ in $a_4$.
22. $(a_1 b_2)$, $b_3$ in $a_2$, $b_4$ in $a_3$, $b_1$ in $a_4$.

| | | |
|---|---|---|
| Index | 4 | $(x+1)(x^4-1)$. |
| | 6 | $(x^2-1)(x+1)(x^2-x+1)$. |
| | 10 | $(x^2-1)(x^5+1)$. |
| | 18 | $(x^2-1)(x^6-x^3+1)$. |
| | 8 | $(x-1)(x^2+1)(x^4+1)$. |
| | 14 | $(x-1)(x^7+1)$. |
| | 30 | $(x^2-1)(x^2-x+1)(x^4-x^3+x^2-x+1)$. |

Unter den Potenzen erwähne ich insbesondere die beiden involutorischen Transformationen, welche ebenfalls typisch sind:

1. $(4, 4, 4, 4, 4, 4)^7$. Dieselbe existirt und besteht aus den Punktepaaren, welche 6 gegebene Punkte zu 8 Basispunkten eines Netzes von $M_2^3$ ergänzen. Char. Function $(x-1)(x+1)^6$. Die Transformation verwandelt das Stralensystem der Bisekanten von $C_3(1\ 1\ 1\ 1\ 1\ 1)$ in sich.*

2. $(8, 8, 8, 8, 8, 8, 8)^{15}$. Dieselbe existirt und besteht aus den Punktepaaren $pp'$ der Eigenschaft, dass eine $M_2^4$ mit 7 festen Doppelpunkten, welche durch $p$ oder $p'$ geht, nothwendig auch $p'$ oder $p$ enthält. Char. Function $(x-1)(x+1)^7$.

Für diese beiden Transformationen sind die char. Functionen der Substitutionen (1) und (2) gleich und hieraus folgt die Gleichheit bei jedem der Typen 16.–22. Es wird ferner gleichwie in §3 bewiesen, dass die Substitutionen (2) für $(a_1 b_1)$ sich von denen für die Reste nur um den Factor $x-1$ unterscheiden und für $(a_1 b_1)$ folgt also jene Gleichheit; hieraus wegen der Symmetrie auch für $(a_1 b_2)(a_2 b_1)$. Da ferner die allgemeinen Typen durch Hinzunahme gewöhnlicher Cyclen entstehen und diese für (1) und (2) dieselben sind, so gilt:

*Theorem LVII.—Für alle periodischen cubischen Reciprokaltransformationen sind die characteristischen Functionen der Substitutionen* (1) *und* (2) *gleich.*

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* Die Transformation 7. Ordnung ist von Sturm bei seinen Untersuchungen über das Flächennetz 2. O. sowie Math. Ann. I. bei ganz anderer Gelegenheit angetroffen worden, jene 15. Ordnung dürfte vollständig neu sein.

Aus der Entstehung der beiden involutorischen Typen von oben folgt noch (cf. Acta Math. XIX, p. 169):

*Theorem LVIII.—Die involutorischen Typen* $(4, \ldots . 4)^7$, $(8, \ldots . 8)^{15}$ *sind mit allen Transformationen über* 6, 7 *Fundamentalpunkten vertauschbar.*

Das Theorem gilt aber nicht nur für Transformationen, sondern auch für Characteristiken.*

## §8.—*Die Reciprokaltransformationen, welche eine Raumcurve $M_1^4$ in sich transformiren.*†

I. $\mathfrak{r}_i = 1$, $\mathfrak{y}_{ik} = 0$. *Elliptische Curven.* 1. Die Parameterdarstellung habe $u_1 + u_2 + u_3 + u_4 \equiv 0 \ldots . 1$ als Schnittpunkttheorem. Es sei $u' + mu \equiv \gamma$, wo $m = 1, \ -1, i, \ +\varepsilon, \ -\varepsilon$ sein kann. Zu zwei mit $a_1, a_2$ complanen Punkten $u_1, u_2$ gehören zwei mit $b_1, b_2$ complane Punkte $u_1', u_2'$, aber aus

$$a_1 + a_2 + u_1 + u_2 \equiv 0, \ b_1 + b_2 + u_1' + u_2' \equiv 0, \text{ also } b_1 + ma_1 + b_2 + ma_2 \equiv -2\gamma$$

und den analogen folgt $b_i + ma_i \equiv -\gamma + \omega$, $(i = 1, \ldots . 4)$, wo $2\omega \equiv 0 \ldots . 2)$, sowie $\sum b + m \sum a \equiv -4\gamma$. Da dem Punkte $-(a_1 + a_2 + a_3)$ der Punkt $b_4$ entspricht, also $b_4 - m(a_1 + a_2 + a_3) \equiv \gamma$, so folgt unter Benützung von 2) $m \sum a \equiv -2\gamma + \omega$, $\sum b \equiv -2\gamma + \omega \ldots . 3)$. Diese Bedingungen sind nothwendig und hinreichend, damit die Ebenen durch $a_i a_k$ und $b_i b_k$ die Correspondenz durch projective Büschel aufnehmen, in welchen die Ebenen $a_i a_k a_l$, $a_i a_k a_m$ den Ebenen $b_i b_k b_m$, $b_i b_k b_l$ entsprechen, also damit $(a_i; b_i)^3$ mit $u' + mu \equiv \gamma$ in $M_1^4$ bestehe. Bemerkt sei noch, dass wenn $-(b_i + b_k + b_l)$ mit $\beta_m$ bezeichnet wird und $-(a_i + a_k + a_l)$ mit $\alpha_m$, $\beta)$ auch geschrieben werden kann $b_m - \beta_m \equiv m(a_m - \alpha_m) \equiv -2\gamma + \omega$. Also:

*Theorem LIX.—Die Haupttetraeder b (oder a) der* $(a_i; b_i)^3$, *welche eine* $M_1^4$ *in sich überführen (mit* $\mathfrak{r}_i = 1$, $\mathfrak{y}_{ik} = 0$) *bilden* 4 *Involutionen* 3. *Stufe.*‡

Unter je zwei coordinirten Hauptpunkten $a_i$, $b_i$ bestehen stets vier Correspondenzen H derselben Art wie die gegebene Γ.

2. *Theorem LX.—Die sämmtlichen* $\infty^3$ *Transformationen, welche eine und dieselbe Correspondenz enthalten, bilden in jedem Falle ein lineares* $\infty^3$*-System im* $R_3$.§

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*Cf. auch das citirte Buch (Berlin, Mayer & Müller, 1895).

† Die Anzahl der Cyklen in einer allgemeinen $(a_i, b_i)^3$ habe ich C. R. de Paris 17 Mai 1881 berechnet.

‡ Die Hauptpunkte und die Schnittpunkte der Gegenebenen mit $M_1^4$ sind für alle Transformationen Punktepaare von vier Correspondenzen der Art $u' - u \equiv \gamma$.

§ Dieses wichtige Theorem gilt auch, wenn die beiden correspondirenden elliptischen Gebiete in zwei verschiedenen $M_1^4$ enthalten sind.

Es sei $pp'$ ein Paar entsprechender Punkte des $R_3$. Das Bündel von Ebenen $E$ durch $p$ schneidet $M_1^4$ in einer $I_4^3$, welche sich in eine eben solche Involution überträgt. Es handelt sich zu finden: Wenn von den 4 Schnittpunkten der $M_1^4$ mit $E$ zwei als $a_i$ genommen werden, zwei als $u$, wie oft geschieht es, dass die beiden je in H und Γ correspondirenden Punktepaare vier Punkte in einer Ebene $E'$ sind, welche aber durch $p'$ geht. Die Einhüllende der Ebenen $E'$ ist von der 6. Classe. Denn ist $q'$ ein Punkt von $M_1^4$ und entspricht er dem Punkte $q$ in Γ, so schneidet das Büschel von $E$ durch $pq$ auf $M_1^4$ eine $I_3^1$, welcher durch H eine $I_3^1$ entspricht, deren Tripel also in Ebenen durch eine einfache Sekante von $M_1^4$ sind und deren Verbindungsgeraden eine $M_2^5$ erfüllen durch $(M_1^4)^2$, sodass die $M_2^5$ die Gerade $p'q'$ in 3 freien Punkten schneiden und also durch $q'$ drei Ebenen $E'$ gehen, für welche $q'$ kein Punkt $b$ ist. Als Punkt $b$ bestimmt $q'$ durch H einen Punkt $a$ und dann, weil aus $a$ und $b$ in zwei $C_3$ projicirt wird und die Projectionspunkte von $p$ und $p'$ eine einzige $O^2$ bestimmen (nach dem Theoreme für die Ebene*) drei Ebenen $E_1'$ an welchen er als $b$ theilnimmt. Dies gibt im Ganzen 6. Da aber jede Transformation 6 solche Ebenen $E'$ absorbirt, so resultirt die Zahl 1. Hieraus folgt auch:

*Theorem LXI.—Es gibt ein lineares $\infty^5$ System von Correlationen, welche alle Punktepaare der Correspondenz Γ als Paare conjugirter Punkte enthalten.*

3. Soll der Transformirte $b_i'$ von $b_i$ mit $a_i$ coincidiren, so wird $b(1-m) \equiv -2\gamma+\omega$; es gibt also 4, $\infty$ oder 0, 2, 3, 1 solcher Punktepaare (je nach $m$). Hieraus ist ersichtlich:

*Theorem LXII.—Nur für $u'-u\equiv\gamma$ mit $b_i-a_i\equiv-\gamma$ gibt es Transformationen $b_1$ in $a_1$, $b_2$ in $a_2$, $b_3$ in $a_3$, $b_4$ in $a_4$, welche die Punkte $a$, $b$ in $M_1^4$ haben u. zw. $\infty^3$.*

Ebenso liefert die Voraussetzung $b_i = a_i$ das

*Theorem LXIII.—Nur für $u'+u\equiv\gamma$ gibt es eine Transformation $(a_1 b_1)$, $(a_2 b_2)$, $(a_3 b_3)$, $(a_4 b_4)$ und für $u'-u\equiv\omega$ gibt es $\infty^2$. Für $u'-u\equiv\gamma$ mit $(h-1)\gamma+\omega\equiv 0$ gibt es $\infty^3$ Transformationen $b_i$ in $b_i'$ .... $b_i^h = a_i$ $(i=1,\ldots. 4)$.*

Für jene $b_i$, welche $b_i''$, $b_i'''$, $b_i^{IV}$, oder $b_i^{V} = q_i$ liefern, geben die übrigen Γ stets nur 2, 3, 1 oder 0 Werthe, sodass keine Transformation besteht. Die Werthe sind $b(1+m^3)\equiv(m^2-m-1)\gamma+\omega$, $b(1-m^4)\equiv-(m^3-m^2+m+1)\gamma+\omega$, $b(1+m^5)\equiv(m^4-m^3+m^2-m-1)\gamma+\omega$, $b(1-m^6)\equiv-(m^5-m^4+m^3-m^2+m+1)\gamma+\omega$.

* Kantor, Atti dell' Accademia di Torino XXIX, 1893.

Es folgt auch für $u' - u \equiv \gamma$ aus $\sum a \equiv 2\gamma$, $\sum b \equiv -2\gamma$, $\sum a + \sum b \equiv 0$, also:

*Theorem LXIV.—Für $u' - u \equiv \gamma$ bilden die beiden Haupttetraeder jeder enthaltenden Transformation $\gamma$ associirte Punkte.*

4. Es entstehen so zwei Arten von Sehnen der $M_1^4$, von welchen aus eine der 5 $\Gamma$ in eine Correspondenz $\Gamma_1$ selber Art projicirt wird, nämlich Geraden $(a_i b_i)(a_k b_k)$ oder $(a_i b_k)(a_k b_i)$. Dies sind jedoch nicht alle. Die Punktepaare, deren Sehnen $\Gamma$ in $\Gamma_1$ projiciren, bilden eine Correspondenz $u' + u \equiv c$, u. zw. ist $c \equiv -(\gamma + \gamma_1):(1+m)$, wenn $\Gamma$, $\Gamma_1$ waren $u' + mu \equiv \gamma$, $u' + mu \equiv \gamma_1$. Also gibt es keine Sehne für $m \equiv -1$, weil eben $u' - u \equiv \gamma$ in sich transformirt wird, 4, 2, 3, 1 Regelschaaren für $m = 1, i, -\varepsilon, +\varepsilon$.

5. In jedem Falle werden die $\infty^1 M_2^2$ durch $M_1^4$, also auch die 4 Kegel unter einander vertauscht. Die Projectivität unter den $M_2^2$ kann also nur die Indices 1, 2, 3, 4 besitzen. Allgemein müssen die Erzeugenden $a_1 u_1$, $a_1 u_2$ in die Erzeugenden $b_1 u_1'$, $b_1 u_2'$ verwandelt werden und die Kegelerzeugenden in Kegelerzeugende. Nun sind die 4 Kegelerzeugenden

$$a_1 + u_1 \equiv 0,\quad a_1 + u_1 \equiv \frac{k_1}{2},\quad a_1 + u_1 \equiv \frac{k_2}{2},\quad a_1 + u_1 \equiv \frac{k_1 + k_2}{2}$$

und wegen $b_1 + ma_1 \equiv -\gamma + \omega$, $u' + mu \equiv \gamma$, sind die entsprechenden Erzeugenden

$$b_1 + u_1' \equiv \omega,\quad b_1 + u_1' \equiv -m\frac{k_1}{2} + \omega,\quad b_1 + u_1' \equiv -m\frac{k_2}{2} + \omega,$$

$$b_1 + u_1' \equiv -m\frac{k_1 + k_2}{2} + \omega,$$

also für $m = +1, -1, i, +\varepsilon, -\varepsilon$ erhält man, indem statt $k_1$, $k_2$, $k_1 + k_2$ gesetzt wird $2\omega_1$, $2\omega_2$, $2\omega_3$

$$\begin{array}{llll}
\omega, & \omega_1 + \omega, & \omega_2 + \omega, & \omega_3 + \omega \\
\omega, & \omega_1 + \omega, & \omega_2 + \omega, & \omega_3 + \omega \\
\omega, & \omega_2 + \omega, & \omega_1 + \omega, & \omega_3 + \omega \\
\omega, & \omega_3 + \omega, & \omega_1 + \omega, & \omega_2 + \omega \\
\omega, & \omega_3 + \omega, & \omega_1 + \omega, & \omega_2 + \omega.
\end{array}$$

Indem man nun hierwie $\omega \equiv \omega, \omega_1, \omega_2, \omega_3$ setzt, erhält man.

*Theorem LXV.—Jede $(a_i;\, b_i)^8$ mit $u' - u \equiv \gamma$ und $b_i - a_i \equiv -\gamma$ in $M_1^4$ transformirt alle $M_2^2$ des Büschels in sich. Wenn aber $b_i - a_i \equiv -\gamma + \omega$, werden die*

*$M_2^3$ involutorisch vertauscht und die Kegelflächen in zwei involutorische Paare getheilt.*

*Theorem LXVI.—Jede $(a_i; b_i)^3$ mit $u' + u \equiv \gamma$ und $b_i + a_i \equiv -\gamma$ transformirt die $M_2^3$ in sich, die übrigen transformiren wie in LXV.*

*Theorem LXVII.—Wenn $(a_i; b_i)^3$ in $M_1^4$ durch $a_i$, $b_i$ $u' + iu \equiv \gamma$ oder $u' + \varepsilon u \equiv \gamma$ hervor ruft, bleiben unter den $M_2^3$ durch $M_1^4$ resp. zwei Kegelflächen oder eine Kegelfläche und ein Hyperboloid invariant und der Index ist resp. 2 oder 3.*

*Rationale Curven.* 1. Für $M_1^4$ mit Doppelpunkt $d$ sei $u_1 u_2 u_3 u_4 \equiv k$ des Schnittpunkttheorem und $Bu + Cu' = 0$ die Projectivität. Aus $a_1 a_2 u_1 u_2 = k$ und $b_1 b_2 B^2 u_1 u_2 : C^2 = k$ folgt $b_1 b_2 B^2 = a_1 a_2 C^2$ und $b_2 : b_3 = a_2 : a_3$, oder die 4 Hauptpunkte $a$ haben dasselbe Doppelverhältnis wie die $b$ und ferner $b_i : a_i = \pm C : B$. Wegen $k^3 B : a_1 a_2 a_3 + Cb_4 = 0$ folgt $a_1 a_2 a_3 a_4 = \mp k B^2 : C^2$, $b_1 b_2 b_3 b_4 = \mp k C^2 : B^2$. Die coordinirten Hauptpunktepaare sind also entsprechende Paare in einer von zwei Projectivitäten mit 2 Doppelpunkten in $d$. In jedem Falle bilden die beiden Haupttetraeder 8 associirte Punkte. Die Tetraeder bilden zwei $I_4^3$, in welchen $d$ vierfach vorkommt. Die $\infty^3$ Transformationen bilden ein lineares System. Die $M_2^3$ werden in sich oder für das Zeichen — involutorisch transformirt.

Sei $uu' = C$ die Projectivität. Aus $u_1 u_2 a_1 a_2 = k$, $u_1' u_2' b_1 b_2 = k$ folgt $a_1 b_1 a_2 b_2 C^2 = k^2$, also $a_i b_i = \pm k : C$. Die Hauptpunktepaare bilden zwei Involutionen mit einem Paare in $d$. Aus $kb_4 = Ca_2 a_3 a_4$ folgt $a_1 a_2 a_3 a_4 = \pm k^3 : C^2$, $b_1 b_2 b_3 b_4 = \pm k^2 : C^2$. Nur für 4 bestimmte Werthe von $C$ sind die coordinirten Tetraederpaare je 8 associirte Punkte. Alle $M_2^2$ werden in sich transformirt oder für das Zeichen—(ebenso die 2 Kegeln) involutorisch vertauscht.

2. Für $M_1^4$ mit Spitze $d$ sei $\sum u = 0$ das Schnittpunktetheorem und $u' + xu = c$ die Projectivität. Aus $b_1 + b_2 + x(a_1 + a_2) = -2c$ folgt $b_i + xa_i = -c$. Aus $b_1 - x(a_2 + a_3 + a_4) = c$ folgt $x\sum a = 2c$, $\sum b = -2c$. Conclusionen wie oben. Die $M_2^2$ werden mit dem halben Index der Projectivität in $M_1^4$ transformirt.

3. Für $M_1^4$ 2. Art muss die $M_2^2$ in sich transformirt werden. Die Trisekanten durch $a_i$ werden in jene durch $b_i$ verwandelt und es sind also jene Punktepaare der cubischen Trisekanteninvolution zu suchen, welche durch die Projectivität in eben solche Punktepaare verwandelt werden. Die Uebertragung auf einen Kegelschnitt lehrt sofort ihre Zahl 4 kennen und es gibt also nur eine Transformation, wenn überhaupt; ausgenommen, wenn die Projectivität die kubische Involution reproducirt. Jene Transformation existirt aber auch stets. Sind nämlich $a_i$, $b_i$ die 3. Stützpunkte der Trisekanten durch jene Punktepaare; so

existiren 2 Ebenen durch $a_1 a_2$, deren Schnittpunktepaare anderen mit $b_1 b_2$ complanen entsprechen und da die quadratische Involution aller durch die letzteren zwei Punktepaare bestimmt ist, so folgt, dass die Projectivität in $M_1^4$ von $a_1 a_2$ und $b_1 b_2$ durch zwei projective Ebenenbüschel aufgenommen wird. Die 6 Ebenenbüschel definiren aber die Transformation.

*Theorem LXVIII.—Es gibt eine einzige $(a_i; b_i)^3$, welche eine $M_1^4$ 2. Art durch $a_i$, $b_i$ in sich transformirt und in ihr eine ganz willkürliche Projectivität hervorruft.*

Wenn aber die cubische Involution in sich verwandelt, dann kann man irgend 3 Punkte als $a$ nehmen, deren Verbindungsgeraden als Axen von Ebenenbüscheln die Transformation und also den 4. Punkt $a$ bestimmen, daher:

*Theorem LXIX.— Wenn die Projectivität, welche in $M_1^4$ 2. Art hervorgerufen werden soll, mit der inhärenten cubischen Involution vertauschbar ist, gibt es $\infty^3$ $(a_i; b_i)^3$, welche ein lineares System bilden.*

In der Aufzählung dieser Projectivitäten erscheint zuerst eine Involution, welche für jede $M_1^4$ die vier Doppelelemente in zwei Paare trennt (eingeschlossen Coincidenz 2[er] Doppelelemente), dann die Fälle, wo sie ein harmonisches Quadrupel (eingeschlossen die Coincidenz 3[er]) oder ein äquianharmonisches bilden und die Indices 2, 4, 3 erscheinen, endlich der Fall, wo zweimal zwei Doppelelemente coincidiren, was die Curve mit zwei stationären Tangenten ist und $\infty^1$ Projectivitäten die Involution in sich verwandeln.

Ein sehr einfaches Verfahren, um die Transformation zu construiren, ist indessen in allen diesen Fällen das in XIII angewandte.* Denn die ebene birationale Transformation, welche die Projection der $M_1^4$ in sich überführt, bestimmt die räumliche $(a_i; b_i)^3$ vollkommen.

II. *Die übrigen Singularitätencomplexe.* $\mathfrak{x}_i = 0$, $\mathfrak{y}_{12} = \mathfrak{y}_{23} = \mathfrak{y}_{34} = \mathfrak{y}_{41} = 2$, $\mathfrak{y}_{13} = \mathfrak{y}_{24} = 0$. Seien $\alpha_1\alpha_2$, $\alpha_3\alpha_4$, $\alpha_5\alpha_6$, $\alpha_7\alpha_8$ die Treffpunkte von $M_1^4$ mit $a_1 a_2$, $a_2 a_3$, $a_3 a_4$, $a_4 a_1$ und analog $\beta_1 \dots \beta_8$ für die $b_i b_k$. Dann folgt aus $\alpha_1 + \alpha_2 + u_1 + u_2 \equiv 0$, $\beta_1 + \beta_2 + u_1' + u_2' \equiv 0$; $\alpha_1 + \alpha_2 + m\beta_1 + m\beta_2 \equiv -2\gamma$, wenn $u' + mu \equiv \gamma$ die Correspondenz. Hieraus und aus der analogen $\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + m(\beta_1 + \beta_2 + \beta_1 + \beta_6) \equiv -4\gamma$, daher $4\gamma \equiv 0$. Es entspreche die Tangentenebene in $\alpha_2$, welche $\alpha_1$ enthält, der Ebene $\beta_1\beta_2\beta_5$; dann folgt $-(\alpha_1 + 2\alpha_2)$

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* Das Verfahren ist im Wesen eine strikte Verallgemeinerung des in der Preisschrift und in §8 hier angewandten, indem nur das binäre Gebiet der invarianten Curve durch das ternäre Gebiet der invarianten Fläche ersetzt wird. Die Abbildung erscheint dabei als Hilfsconstruction, welche den Parametercalcül ersetzt.

$-m(\beta_3+\beta_2)-m\beta_5\equiv\gamma$ und $\gamma\equiv\alpha_2+\beta_5\equiv\alpha_1+m\beta_6$ sowie $\gamma\equiv\alpha_3+m\beta_7$, $\gamma\equiv\alpha_4+m\beta_8$. Hieraus $\alpha_1+\alpha_2\equiv\alpha_5+\alpha_6$, $\beta_1+\beta_2\equiv\beta_5+\beta_6$ und $\alpha_5+\alpha_6 \equiv\alpha_7+\alpha_8$, $\beta_3+\beta_4\equiv\beta_7+\beta_8$. Durch $4\gamma\equiv 0$ und die letzten 8 Bedingungen ist alles bestimmt. Es können also $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_8$ mit den Bedingungen $\alpha_1+\alpha_2\equiv\alpha_5+\alpha_6$, $\alpha_3+\alpha_4\equiv\alpha_7+\alpha_8$ willkürlich angenommen werden. Für die Curve mit Spitze gelten natürlich dieselben Rechnungen mit Gleichungen. Für $M_1^4$ mit Doppelpunkt kommt $C\beta_5=-B\alpha_2$, $C\beta_6=-B\alpha_1$, $C\beta_7=-B\alpha_3$, $C\beta_8=-B\alpha_4$, $\alpha_1\alpha_2=\alpha_5\alpha_6$, $\beta_1\beta_2=\beta_5\beta_6$, $\alpha_3\alpha_4=\alpha_7\alpha_8$, $\beta_3\beta_4=\beta_7\beta_8$. Auch für die $M_1^4$ 2. Art sind $a_1 \ldots\ldots a_8$ 8 associirte Punkte, weil zwei Ebenenpaare und die $F_2$ der $M_1^4$ hindurchgehen. Folglich muss die Correspondenz ebenfalls eine collineare sein. Von diesen ist oben in I. gesprochen worden.* Für jede solche Collineation kann $a_1 a_2$ und sie schneidend $a_3 a_4$ und $a_7 a_8$ willkürlich genommen werden. Es gibt $\infty^4 (a_i;\, b_i)^3$ für jede Collineation.

$\mathfrak{r}_i=0$, $\mathfrak{y}_{12}=\mathfrak{y}_{34}=2$, $\mathfrak{y}_{23}=\mathfrak{y}_{31}=\mathfrak{y}_{24}=\mathfrak{y}_{41}=1$. Seien $\alpha_1\alpha_2$, $\alpha_3\alpha_4$ die Treffpunkte mit $a_1a_2$, $a_3a_4$; $\alpha_5$, $\alpha_6$, $\alpha_7$, $\alpha_8$ jene mit $a_1a_3$, $a_1a_4$, $a_2a_3$, $a_2a_4$ und $\Gamma$ sei $u'+mu\equiv\gamma$. Aus $\alpha_1+\alpha_2+u_1+u_2\equiv 0$, $\beta_1+\beta_2+u_1'+u_2'\equiv 0$ folgt $\alpha_1+\alpha_2 + m(\beta_1+\beta_2)\equiv -2\gamma$, analog $\alpha_3+\alpha_4+m(\beta_3+\beta_4)\equiv -2\gamma$. Aus $\alpha_5+u_1 +u_2+u_3\equiv 0$, $\beta_5+u_1'+u_2'+u_3'\equiv 0$ folgt $\alpha_5+m\beta_5\equiv -3\gamma$, $\alpha_6+m\beta_6\equiv -3\gamma$, $\alpha_7+m\beta_7\equiv -3\gamma$, $\alpha_8+m\beta_8\equiv -3\gamma$, ferner $\alpha_1+\alpha_2+\alpha_5+\alpha_7+m(\beta_1+\beta_2+\beta_5+\beta_7) \equiv -8\gamma$, also $8\gamma\equiv 0$, $\alpha_5-\alpha_7\equiv\alpha_6-\alpha_8\equiv 4\gamma$. Durch $\Gamma$ wird nicht jede $I_3^1$ eines Ebenenbüschels wieder in eine $I_3^1$ eines Ebenenbüschels übergeführt. Man muss also zu $a_1a_2$, $a_3a_4$ noch eine ausgewählte Axe hinzunehmen, sodass es $\infty^5 (a_i;\, b_i)^3$ gibt.

$\mathfrak{r}_1=1$, $\mathfrak{r}_2=\mathfrak{r}_3=\mathfrak{r}_4=0$, $\mathfrak{y}_{ik}=1$. Sind $a_2 a_7$ die Punkte auf $a_1a_2$, $a_1a_3$, $a_1a_4$, $a_3a_4$, $a_4a_2$, $a_2a_3$ und entsprechend die $\beta$, so werden $b_1\beta_2\beta_3\beta_4$ stets ein Haupttetraeder für eine $(a_i;\, b_i)^3$ aus I dieses § bilden und es ist zu fragen, ob die vier Schnittpunkte von $M_1^4$ mit den Seitenebenen von $a_1a_2a_3a_4$ in einer Ebene sind also $6\gamma+\omega\equiv 0$ ist. Die Construction geschieht wieder durch drei Ebenenbüschel.

$\mathfrak{r}_1=1$, $\mathfrak{r}_2=\mathfrak{r}_3=\mathfrak{r}_4=0$, $\mathfrak{y}_{13}=\mathfrak{y}_{14}=\mathfrak{y}_{34}=\mathfrak{y}_{23}=1$, $\mathfrak{y}_{24}=2$; $\mathfrak{r}_4'=1$, $\mathfrak{y}_{14}'=\mathfrak{y}_{23}'$

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*** Diese Collineationen sind leicht zu bilden, da sie alle die $M_2^2$ in sich transformiren. Die Aufzählung bei Brambilla, Atti Ist. Veneto XIX, "Le omografie, che mutano in se stessa una curva gobba di 4° ordine" ist unvollständig.**

$= \mathfrak{y}'_{12} = \mathfrak{y}'_{24} = 1, \mathfrak{y}'_{13} = 2$. Wenn $\alpha_1, \ldots \alpha_6$ auf $a_1 a_3$, $a_1 a_4$, $a_3 a_4$, $a_2 a_3$, $a_2 a_4$ sind und entsprechend $\beta_i$, folgen $\alpha_1 + m\beta_1 \equiv \gamma, \ldots \alpha_6 + m\beta_6 \equiv \gamma$, woraus $\alpha_3 - \alpha_1 \equiv 4\gamma$. Man nehme $a_1$, $\alpha_1$ an, wodurch $\alpha_3$ folgt, ziehe die Ebene $a_1 \alpha_1 \alpha_3$, welche $M_1^4$ in $\alpha_2$ schneidet, definire $b_4$ aus $\alpha_2$ mittelst $\alpha_2 + mb_4 \equiv -3\gamma$, ziehe durch $b_4$ jene Sehne, welche der $a_1 a_2$ entspricht, welche die der $a_1 \alpha_1$ entsprechende Regelschaar in einem Punkte trifft. Von diesem als $b_1$ ziehe man den Stral der Regelschaar als $\beta_5 \beta_6$, welcher die dem Stralbüschel um $\alpha_3$ in $\alpha_3 a_1 \alpha_1$ entsprechende Ebene in $b_3$ schneidet. Hiemit sind drei Paare projectiver Ebenenbüschel bestimmt.

$\mathfrak{x}_1 = \mathfrak{x}_2 = 1$, $\mathfrak{y}_{13} = \mathfrak{y}_{24} = 1$, $\mathfrak{y}_{34} = 2$; $\mathfrak{x}'_3 = \mathfrak{x}'_4 = 1$, $\mathfrak{y}'_{13} = \mathfrak{y}'_{24} = 1$, $\mathfrak{y}'_{12} = 0$. Man nehme $a_1$ an, bestimme hierauf $b_4$ gemäss der Bedingung des Entsprechens der cubischen Involutionen für $a_1 a_4$ und $b_1 b_4 (a_1 + mb_4 \equiv -3\gamma)$, ziehe $a_1 \alpha_3$ und die entsprechende Regelschaar, ziehe $\alpha_3 \alpha_4$ schneidend $a_1 \alpha_1$ und bestimme die Regelschaar, hierauf das Stralbüschel durch $b_4$ entsprechend dem Stralbüschel um $a_1$ in $a_1 a_3 a_4$, durch $b_4$ die Erzeugende der 1. Regelschaar, welche $b_3$ bestimmt, durch $b_3$ die Erzeugende der 2. Regelschaar, welche $M_1^4$ in $\beta_2$ und die Ebene durch $b_4$ in $b_1$ trifft. Hiemit sind 3 Paare projectiver Ebenenbüschel bestimmt.

$\mathfrak{x}_1 = \mathfrak{x}_2 = 1$, $\mathfrak{y}_{13} = \mathfrak{y}_{14} = \mathfrak{y}_{23} = \mathfrak{y}_{24} = 1$, $\mathfrak{y}_{34} = 0$; $\mathfrak{x}'_1 = \mathfrak{x}'_2 = 1$, $\mathfrak{y}'_{13} = \mathfrak{y}'_{14} = \mathfrak{y}'_{23} = \mathfrak{y}'_{24} = 1$. Die Punkte $a_1 a_2 \alpha_1 \alpha_2$ sind Punkte eines Haupttetraeders, $a_2 a_4 \alpha_3 \alpha_4$ ebenfalls und daher gilt $a_1 + a_2 + \alpha_1 + \alpha_2 \equiv a_1 + a_2 + \alpha_3 + \alpha_4$ und $a_1 + a_2 + \alpha_1 + \alpha_2 \equiv a_1 + a_2 + \alpha_3 + \alpha_4 \equiv 0$. Hienach bilden die vier $\alpha$ zwei Paare conjugirter Punkte, während $a_1$, $a_2$ willkürlich sind. Hiemit ist die Construction vollendet.

$\mathfrak{x}_1 = \mathfrak{x}_2 = 1$, $\mathfrak{y}_{13} = \mathfrak{y}_{23} = \mathfrak{y}_{24} = \mathfrak{y}_{34} = 1$; $\mathfrak{x}'_2 = \mathfrak{x}'_3 = 1$, $\mathfrak{y}'_{13} = \mathfrak{y}'_{12} = \mathfrak{y}'_{14} = \mathfrak{y}'_{24} = 1$. Man nehme $a_1$, $a_2$, ziehe aus $a_1$, $a_2$ Sehnen $a_1 \alpha_1$, $a_2 \alpha_2$, welche sich schneiden, wodurch drei Regelschaaren bestimmt werden und es muss in der 3. eine Gerade genommen werden, welche $M_1^4$ in $b_2$, $b_3$ derart trifft, dass die von $b_2$, $b_3$ in die 1., resp. in die 2. Regelschaar gezogenen Geraden sich schneiden. Diese drei Geraden sind die Axen dieser zu den ersteren projectiven Ebenenbüschel.

Für $M_1^4$ 2. Art ist die Construction der Fälle, wo nothwendig Ebenenbüschel mit einpunktigen Sekanten als Axen zu verwenden sind, schwieriger, weil nicht jede cubische Involution durch ein Ebenenbüschel ausgeschnitten wird. Dagegen kommen für sie einige Singularitätencomplexe hinzu: $\mathfrak{x}_1 = \mathfrak{x}_2 = 1$, $\mathfrak{y}_{13} = \mathfrak{y}_{24} = 2$; $\mathfrak{x}'_1 = \mathfrak{x}'_2 = 1$, $\mathfrak{y}'_{13} = \mathfrak{y}'_{24} = 2 \,|\, \mathfrak{x}_1 = \mathfrak{x}_2 = 1$, $\mathfrak{y}_{34} = 3$, $\mathfrak{y}_{12} = 1 \,|\, \mathfrak{x}_1 = 1$, $\mathfrak{y}_{12} = 2$, $\mathfrak{y}_{34} = 3$, $\mathfrak{y}_{13} = 1 \,|\, \mathfrak{x}_1 = \mathfrak{x}_2 = 1$, $\mathfrak{y}_{12} = \mathfrak{y}_{13} = 1$, $\mathfrak{y}_{34} = 2 \,|\, \mathfrak{x}_1 = \mathfrak{x}_2 = 1$, $\mathfrak{y}_{12} = \mathfrak{y}_{13} = 1$, $\mathfrak{y}_{14} = 2 \,|\, \mathfrak{x}_1 = \mathfrak{x}_2 = \mathfrak{x}_3 = 1$, $\mathfrak{y}_{14} = 2 \,|\, \mathfrak{x}_1 = \mathfrak{x}_2 = \mathfrak{x}_3 = 1$, $\mathfrak{y}_{12} = 1$, $\mathfrak{y}_{14} = 1 \,|\, \mathfrak{x}_1 = \mathfrak{x}_2 = \mathfrak{x}_3 = 1$, $\mathfrak{y}_{12} = 1$, $\mathfrak{y}_{34} = 1$.

## §9.—*Die aperiodischen Transformationen $b_i$ in $a_k$.*

Da sich in der Ebene an die Figur von ($a'$ in $a$, $b'$ in $b$, $c'$ in $c$)$^{3}$ so bedeutendes Interesse knüpft, will ich wenigstens über die Existenz der genannten $(a_i;\, b_i)^3$ entscheiden, bevor ich zu den Typen übergehe.

Aus den 8 Bedingungen $b_i + ma_i \equiv -\gamma + \omega$, $a_k + mb_i \equiv \gamma$ folgt $\sum b + m\sum a \equiv -4\gamma$ und $m\sum b + \sum a = 4\gamma$, $\sum b\,(1-m^2) \equiv -4\,(1+m)\gamma$, und wegen $\sum b \equiv -2\gamma + \omega$ (§8), $2\,(1+m)^2\,\gamma \equiv \omega\,(m^2-1)$, woraus für $m = 1 \ldots .\ 8\gamma \equiv 0$; $m = -1, \ldots .$ keine Bedingung; $m = i \ldots .\ 4\gamma \equiv 0$; $m = +\varepsilon \ldots .\ 2\gamma \equiv \omega$, also $\sum b \equiv 0$; $m = -\varepsilon \ldots .\ 6\gamma \equiv \omega$ folgt. $u' + \varepsilon u \equiv \gamma$ ist also nicht in Betracht zu ziehen, und für $u' + iu \equiv \gamma$ ist $\Gamma$ collinear.

Für $M_1^4$ mit Spitze folgt aus $b_i + xa_i = -c$, $a_k + xb_i = c \ldots .\ \sum b + x\sum a = -4c$, $\sum a + x\sum b = 4c$, $\sum a\,(1-x^2) = 8c$, $\sum b\,(1-x^2) = -4c\,(1+x)$, und wegen $\sum b = -2c$, $x^2 = -1$, die Rechnung ist also dieselbe wie für $u' + iu \equiv \gamma$.

I. $b_i$ *in* $a_i$ $(i = 1, \ldots .\ 4)$. Aus $u' + u \equiv \gamma$ würde folgen $2\gamma + \omega \equiv 0$, also $\sum b \equiv 0$, Collineation. Aus $u' - \varepsilon u \equiv \gamma$ folgt $b_i\,(1-\varepsilon^2) \equiv (\varepsilon - 1)\,\gamma + \omega$, $a_i\,(1-\varepsilon^2) \equiv (1-\varepsilon)\,\gamma + \varepsilon\omega$, woraus hervorgeht, dass Coincidenz irgend eines $b$ mit einem $a$ stattfinden müsste. Aus $u' - u \equiv \gamma$ folgt wie in §8, dass es $\infty^3$ Transformationen gibt. Aus $a_i + a_k + b_l + b_m \equiv 0$ (§8) folgt: von den beiden Haupttetraedern schneidet jede Kante die der entsprechenden gegenüberliegende des anderen Tetraeders. Dies gibt bereits 6 uneigentliche Doppelpunkte. Die drei Ebenenpaare des Netzes von $M_2^3$ (Th. LXIV) geben drei Doppelgerade, welche eine $M_1^3$ zur Kegelspitzencurve ergänzt. Diese $M_1^3$ ist der Ort der Doppelpunkte der Transformation, geht durch jene 6 uneigentlichen Doppelpunkte und schneidet die 4 Geraden $a_i b_i$ in je einem Punkte. Aus $u' + iu \equiv \gamma$ folgt $b_i \equiv -\dfrac{(1+i)\gamma}{2} + \dfrac{\omega}{2}$, $\sum b \equiv -2\gamma + \omega$, also $3\gamma \equiv -i\omega$, es darf also weder $\gamma$ noch $\omega$ gleich 0 oder $\dfrac{k_1 - k_2}{2}$ sein. Nun ist ferner

$$a_k + a_l + b_k + b_l \equiv a_k + a_l + b_m + b_n \equiv a_m + a_n + b_k + b_l$$
$$\equiv a_m + a_n + b_m + b_n \equiv \omega\,(1-i) + (1-i)_\omega,$$

sodass also von zwei Paaren Kanten $a_i a_k$ jedes das entsprechende Paar Kanten $bb$ schneidet. Zwei Geraden $a_k b_k$, $a_l b_l$ gehen durch die eine Kegelspitze des Büschels, zwei andere durch eine andere, seien es $a_m b_m$, $a_n b_n$. Dann gehen auch $a_m b_n$, $a_n b_m$ durch die erste, $a_k b_l$, $a_l b_k$ durch die 2. Kegelspitze. Die beiden Tetraeder sind perspectivisch und da sie associirt sind, auch desmisch. Da zwei desmische Tetraeder in irgend zwei andere übergeführt werden können, folgt

*Theorem LXX.—Zwei desmische* Tetraeder, in der Weise als Haupttetraeder genommen, dass $a_1 b_1$, $a_2 b_2$, $a_3 b_4$, $a_4 b_3$ sich in einer Ecke des 3. Tetraeders schneiden, gestatten eine $(a_i; b_i)^3$ mit der Characteristik $b_i$ in $a_i$ $(i = 1, \ldots, 4)$. Es gibt stets eine in sich transformirte harmonische $M_1^4$.*

Im Falle $u' - u \equiv \gamma$ können die beiden Tetraeder ebenfalls desmisch werden, aber nur dann, wenn $2a_i \equiv 2\gamma$ wird und $2b_i \equiv -2\gamma$, es wird $b_1 \equiv -a_1 \equiv -\frac{\gamma}{2} + \omega'$, daher:

*Theorem LXXI.—Zwei desmische Tetraeder, in der Weise als Haupttetraeder genommen, dass $a_i b_i$ sich in einer Ecke des 3. Tetraeders schneiden, gestatten eine $(a_i; b_i)^3$ mit der Characteristik $b_i$ in $a_i (i = 1, \ldots 4)$. Alle $M_2^2$ des Netzes werden in sich transformirt.*

II. *$b_1$ in $a_1$, $b_2$ in $a_2$, $b_3$ in $a_4$, $b_4$ in $a_3$.* Für $u' + u \equiv \gamma$ folgt $2\gamma + \omega \equiv 0$, $\sum b \equiv 0$, also Unmöglichkeit; für $u' - u \equiv \gamma$ folgt aus $b_3 - a_3 \equiv -\gamma + \omega$, $a_3 - b_4 \equiv \gamma$ wegen $\omega \equiv 0$, $b_3 \equiv b_4$ also die vorige Characteristik; für $u' - \varepsilon u \equiv \gamma$ folgt $b_3 - \varepsilon^2 b_4 \equiv (\varepsilon - 1)\gamma + \omega$, $b_4 - \varepsilon^2 b_3 \equiv (\varepsilon - 1)\gamma + \omega$, $b_3 - b_4 \equiv 0$, also ebenfalls I. Für $u' + iu \equiv \gamma$ folgt $(b_3 - b_4)(1 + i) \equiv 0$, $b_3 \equiv b_4 + \frac{k_1 - k_2}{2}$. Ferner wird $b_3 + b_4 \equiv -(1 + i)\gamma + \omega$, $2b_1 \equiv 2b_2 \equiv -(1 + i)\gamma + \omega$, also $2(b_3 + b_4) \equiv 2(b_1 + b_2) \equiv 0$. Es gehen $a_1 a_2$, $b_1 b_2$, $a_3 a_4$, $b_3 b_4$ durch Kegelspitzen. Es hängt jedoch von den Werthen $\omega$, $b_1$ ab, ob diese Kegelspitzen theilweise dieselben sind. Wenn $\omega(1 - i) \equiv 0$ und $b_1 + b_2 \equiv -\gamma(1 + i) + \omega$, so schneiden $a_1 a_2$ und $a_3 a_4$ die $b_1 b_2$ und $b_3 b_4$. Hiebei ist $a_3$ noch willkürlich, bestimmt aber dann $b_3$, $a_4$, $b_4$. In der Weise des Theoremes LXXI können sie jedoch nicht zwei desmische Tetraeder bilden. Denn man hat:

$$\begin{array}{llll}
1)\ a_1 + a_3 \equiv & a_3 + g - i\varpi - ic & 1')\ b_1 + b_3 \equiv & -ia_3 - g - \gamma + 3\varpi + c \\
2)\ a_3 + a_4 \equiv & -a_3 + 3g - 3i\varpi - ic & 2')\ b_1 + b_4 \equiv & ia_3 - g - i\gamma + \varpi + c \\
3)\ a_1 + a_2 \equiv & g - i\omega - ic & 3')\ b_1 + b_2 \equiv & -2g + \omega + c + c' \\
4)\ a_2 + a_3 \equiv & a_3 + g - i\varpi - ic' & 4')\ b_2 + b_3 \equiv & -ia_3 - g - \gamma + 3\varpi + c' \\
5)\ a_2 + a_4 \equiv & -a_3 + g - 3\varpi - ic' & 5')\ b_2 + b_4 \equiv & ia_3 - g - i\gamma + \varpi + c' \\
6)\ a_3 + a_4 \equiv & 2g - i\omega & 6')\ b_3 + b_4 \equiv & -2g + \omega,
\end{array}$$

* Für die Definition der "desmischen" Configuration verweise ich auf Reye, Acta Math. I. Dort wie in der ganzen bezüglichen Literatur wird durchwegs nur von Collineationen und Correlationen über der Figur gehandelt.

5

wo $g \equiv -\gamma \frac{(1+i)}{2}$, $\varpi \equiv \frac{\omega}{2}$, $c \equiv \frac{C}{2}$, $c' \equiv \frac{C'}{2}$. Es geben nun 1) + 1'), 1) + 5'), 5) + 1'), 5) + 5') simultan unter Beachtung von $(1-i)\omega \equiv 0$, $(1-i)\frac{C+C'}{2} \equiv 0$, $\gamma(1-i) \equiv 0$ und hiemit erscheinen für alle 8 Punkte nur 4 Werthe. Ebenso geben 1) + 2'), 1) + 4'), 2) + 2'), 2) + 4'), $c + c' \equiv 0$, während $c + c' \equiv \frac{k_1 - k_2}{2}$ sein muss. Ferner 1) + 3'), 1) + 6'), 5) + 3'), 5) + 6') geben $c + c' \equiv 0$, also $\Sigma b \equiv 0$.

In dem möglichen Falle also sind die Spitzen der zwei invarianten Kegel die Doppelpunkte von $\Gamma$ und jene auf $a_1 b_1$, $a_2 b_2$ die 8 Doppelpunkte von $(a_i; b_i)^3$, wovon zwei uneigentlich werden können eben für $\omega(1-i) \equiv (1-i)(c+c') \equiv 0$.

III. $b_1$ *in* $a_2$, $b_2$ *in* $a_1$, $b_3$ *in* $a_4$, $b_4$ *in* $a_3$. Aus $u' + u \equiv \gamma$ folgt $4\gamma \equiv 0$, $b_1 - b_2 \equiv -2\gamma + \omega$, $b_3 - b_4 \equiv -2\gamma + \omega$. Es sind $a_1$ und $a_3$ ganz willkürlich, hieraus folgen die übrigen Punkte. Das desmische Tetraeder kann eintreten, wenn $2\gamma \equiv 0$. In der That, man hat:

| | |
|---|---|
| 1) $a_1 + a_2 \equiv 2a_1 + 2\gamma - \omega$ | 1') $b_1 + b_2 \equiv -2a_1 + \omega$ |
| 2) $a_1 + a_4 \equiv a_1 + a_3 + 2\gamma - \omega$ | 2') $b_1 + b_4 \equiv - a_1 - a_3 + \omega$ |
| 3) $a_3 + a_4 \equiv 2a_3 + 2\gamma - \omega$ | 3') $b_3 + b_4 \equiv -2a_3 + \omega$ |
| 4) $a_2 + a_4 \equiv a_1 + a_3$ | 4') $b_2 + b_4 \equiv - a_1 - a_3 + 2\gamma$ |
| 5) $a_2 + a_3 \equiv a_1 + a_3 + 2\gamma - \omega$ | 5') $b_2 + b_3 \equiv - a_1 - a_3 + \omega$ |
| 6) $a_1 + a_3 \equiv a_1 + a_3$ | 6') $b_1 + b_3 \equiv - a_1 - a_3 - 2\gamma$, |

1) + 1'), 2) + 3'), 3) + 1'), 3) + 3') geben $2\gamma \equiv 0$, $2a_1 \equiv 2a_3$; ebenso die übrigen zusammengehörigen Gegenkantenpaare. Also:

*Theorem LXXII.—Zwei desmische Tetraeder, in der Weise als Hauptpunkte genommen, dass $a_i b_i$ durch einen Eckpunkt des 3. Tetraeders gehen, gestatten $(a_i; b_i)^3$ mit der Characteristik III.*

Man konnte dies auch aus I folgern durch Zusammensetzung mit einer der Collineationen, welche in der desmischen Configuration walten.

Aus $u' - u \equiv \gamma$ folgt $4\gamma \equiv 0$, ferner nach Annahme von $a_1$, $a_3$ folgt $b_1 \equiv a_1 - \gamma + \omega$, $b_2 \equiv a_1 - \gamma$, $b_3 \equiv a_3 - \gamma + \omega$, $b_4 \equiv a_3 - \gamma$, $a_2 \equiv a_1 + \omega$, $a_4 \equiv a_3 + \omega$. Hieraus folgt $\sum b \equiv 2a_1 + 2a_3$ also $2a_1 + 2a_3 \equiv -2\gamma + \omega$, weshalb die Kantenpaare sich niemals schneiden können, die desmischen Tetraeder gestatten also diesen Fall nicht. Aus $u' + iu \equiv \gamma$ folgt $b_1 + b_2 \equiv b_3 + b_4 \equiv -\gamma(1+i) + \omega$, $a_1 + a_2 \equiv a_3 + a_4 \equiv \gamma(1+i) - i\omega$, die Formeln

1) $a_1 + a_2 \equiv (i+1)\gamma - i\omega$ 1') $b_1 + b_2 \equiv -(1+i)\gamma + \omega$

2) $a_1 + a_3 \equiv a_1 + a_3$ 2') $b_1 + b_3 \equiv -ia_1 - ia_3 - 2\gamma$

3) $a_1 + a_4 \equiv a_1 - a_3 + (i+1)\gamma - i\omega$ 3') $b_1 + b_4 \equiv -ia_1 + ia_3 - (1+i)\gamma + \omega$

4) $a_3 + a_4 \equiv -a_3 + (i+1)\gamma - i\omega$ 4') $b_3 + b_4 \equiv -(1+i)\gamma + \omega$

5) $a_2 + a_4 \equiv -a_1 - a_3 + 2(i+1)\gamma$ 5') $b_2 + b_4 \equiv ia_1 + ia_3 - 2i\gamma$

6) $a_2 + a_3 \equiv -a_1 + a_3 + (i+1)\gamma - i\omega$ 6') $b_2 + b_3 \equiv ia_1 - ia_3 - (i+1)\gamma + \omega$,

beweisen, dass für $\omega(1-i)$, $a_3 \equiv 0$, $a_1(1-i) \equiv 2\gamma \equiv 0$ zwei desmische Tetraeder entstehen können.

*Theorem LXXIII.—Zwei desmische Tetraeder, in der Weise als Hauptpunkte genommen, dass $a_1 b_1$, $a_2 b_2$, $a_3 b_4$, $a_4 b_3$ durch einen Eckpunkt des dritten gehen, gestatten $(a_i; b_i)^3$ mit Characteristik III.*

Aus $u' - \varepsilon u \equiv \gamma$ folgt sofort $b_3 \equiv b_4$, $b_1 \equiv b_2$, also I.

IV. $b_1$ *in* $a_1$, $b_2$ *in* $a_3$, $b_3$ *in* $a_4$, $b_4$ *in* $a_2$. Für $u' + u \equiv \gamma$ wird $b_2 \equiv b_3 \equiv b_4$; für $u' - u \equiv \gamma$ zunächst $\omega \equiv 0$ und ferner $b_2 \equiv b_3 \equiv b_4$; für $u' + iu \equiv \gamma$ wird $b_2 + b_3 \equiv b_3 + b_4 \equiv b_4 + b_2$, also $b_2 \equiv b_3 \equiv b_4$. Für $u' - \varepsilon u \equiv \gamma$ wird $a_1(1-\varepsilon^2) \equiv -\varepsilon(1-\varepsilon^2)\gamma + \varepsilon\omega$, $b_1(1-\varepsilon^2) \equiv \varepsilon(1-\varepsilon^2)\gamma + \omega$, und

1) $a_1 + a_2 \equiv (1-\varepsilon)\gamma + \varepsilon b_3 + \varepsilon\omega : (1-\varepsilon^2)$ 1') $b_1 + b_2 \equiv (2\varepsilon - 1)\gamma + \varepsilon^2 b_3 + (2-\varepsilon^2)\omega : (1-\varepsilon^2)$

2) $a_1 + a_3 \equiv (\varepsilon^2 - \varepsilon)\gamma + \varepsilon^2 b_3 + (2\varepsilon - \varepsilon^2)\omega : 1 - \varepsilon^2$ 2') $b_1 + b_3 \equiv \varepsilon\gamma + b_3 + \omega : (1-\varepsilon^2)$

3) $a_1 + a_4 \equiv -3\varepsilon\gamma + b_3 + (2\varepsilon - 1)\omega : (1-\varepsilon^2)$ 3') $b_1 + b_4 \equiv (2\varepsilon - \varepsilon^2)\gamma + \varepsilon b_3 + (2-\varepsilon)\omega : (1-\varepsilon^2)$

4) $a_3 + a_4 \equiv (\varepsilon^2 - 2\varepsilon)\gamma - \varepsilon b_3 + (\varepsilon - \varepsilon^2)\omega$ 4') $b_3 + b_4 \equiv (\varepsilon - \varepsilon^2)\gamma - \varepsilon^2 b_3 - \varepsilon\omega$

5) $a_2 + a_4 \equiv (1-2\varepsilon)\gamma - \varepsilon^2 b_3 + \varepsilon\omega$ 5') $b_2 + b_4 \equiv 3\varepsilon\gamma - b_3 + (1-\varepsilon)\omega$

6) $a_2 + a_3 \equiv -\varepsilon\gamma - b_3 - \varepsilon^2\omega$ 6') $b_2 + b_3 \equiv (\varepsilon - 1)\gamma - \varepsilon b_3 + \omega$,

woraus als die gemeinsame Bedingung, damit sich je $a_1 a_2$, $b_3 b_3$; $a_1 a_3$, $b_3 b_4$; $a_1 a_4$, $b_2 b_4$; $a_3 a_4$, $b_1 b_4$; $a_2 a_4$, $b_1 b_2$; $a_2 a_3$, $b_1 b_3$ schneiden, folgt $\rho - 2\varepsilon\omega : (1-\varepsilon^2) \equiv 0$, wo $\rho$ dass in $b_1$ auftretende Periodendrittel, und als die einzige Bedingung, damit sich je $a_1 a_2$, $b_1 b_4$; $a_1 a_3$, $b_1 b_2$; $a_1 a_4$, $b_1 b_3$; $a_3 a_4$, $b_2 b_3$; $a_2 a_4$, $b_3 b_4$; $a_2 a_3$, $b_2 b_4$ schneiden, folgt $2b_3 - 2\varepsilon\gamma - \varepsilon\rho + 2\varepsilon^2\omega : (1-\varepsilon^2) \equiv 0$. Die beiden Bedingungen sind stets verträglich und geben $2b_3 \equiv 2\varepsilon\gamma$, woraus $2(a_i + b_i) \equiv 0$ u. zw. $a_1 + b_1 \equiv -\varepsilon^2\rho - \varepsilon^2\omega : (1-\varepsilon^2)$, $a_2 + b_2 \equiv -b_3 + \varepsilon\gamma + \omega$; $a_3 + b_3 \equiv -\varepsilon b_3 + \varepsilon^2\gamma - \varepsilon^2\omega$; $a_4 + b_4 \equiv -\varepsilon^2 b_3 + \gamma$. Es gehen alle 4 Geraden durch dieselbe Kegelspitze, daher:

*Theorem LXXIV.—Zwei desmische Tetraeder, in der Weise als Haupttetraeder genommen, dass $a_i b_i$ durch einen Eckpunkt des 3. Tetraeders gehen, gestatten $(a_i; b_i)^3$ mit der Characteristik IV.*

Für $M_1^4$ mit Spitze erhält man genau dieselbe Rechnung mit Gleichungen und $\omega = \rho = 0$, sodass die erste Bedingung identisch, die 2. durch $b_3 = \varepsilon c$ befriedigt wird, $b_i + a_i = 0$ wird. Wenn also IV eine $M_1^4$ mit Spitze reproducirt, so schneidet stets jede der 6 Kanten der $a$ eine bestimmte der 6 Kanten der $b$.

V. $b_1$ *in* $a_2$, $b_2$ *in* $a_3$, $b_3$ *in* $a_4$, $b_4$ *in* $a_1$. Für $u' + u \equiv \gamma$ folgt $b_2 \equiv \gamma - a_2$, $b_3 \equiv 3\gamma - a_1 - \omega$, $b_4 \equiv 5\gamma - a_1 - \omega$, $b_1 \equiv 7\gamma - a_1 - \omega$, $a_2 \equiv -2\gamma + a_1 + \omega$, $a_3 \equiv -4\gamma + a_1 + \omega$, $a_4 \equiv -6\gamma + a_1 + \omega$, mit $8\gamma \equiv 0$. Für $\omega \equiv 0$ können also zwar die Schnitte von $a_2 a_4$, $b_3 b_4$; $a_3 a_4$, $b_1 b_3$; $a_1 a_2$, $b_1 b_3$; $a_2 a_3$, $b_2 b_4$; $a_1 a_4$, $b_2 b_4$ erreicht werden, jedoch niemals desmische Tetraeder.

Für $u' - u \equiv \gamma$ folgt $b_1 \equiv b_2$, für $u' + iu \equiv \gamma$ desgleichen. Für $u' - \varepsilon u \equiv \gamma$ folgt zünachst $b_i(1-\varepsilon^2) \equiv (\varepsilon - 1)\gamma + \omega$, sodass jedenfalls für 4 Punkte nur 3 Werthe resultiren. Für $M_1^4$ mit Spitze kommt $x^2 = 1$, und $x = 1$ gibt dieselbe Rechnung wie $u' + u \equiv \gamma$, $x = -1$ gibt $b_1 \equiv b_2$. In dem möglichen Falle sind die 4 Doppelpunkte von $\Gamma$ und die 4 Kegelspitzen die 8 Doppelpunkte von $(a_i; b_i)^3$.

Es ist nunmehr leicht, die $(a_i; b_i)^3$, welche sich auf die desmische Configuration verlegen lassen, vollständig aufzuzählen. Dieselben bilden aber keine endliche Gruppe. Weitere $(a_i; b_i)^3$ in der desmischen Configuration bringt §20.

## §10.—*Die typischen Transformationen mit* $(a_1 b_1)$.

Für die Construction dieser und der folgenden Typen stehen zunächst zwei Methoden zur Verfügung. In der einen baut sich die Transformation über einer in sich transformirten $M_2^2$ auf, deren Feld in bekannter Weise in sich verwandelt wird, aber durch diese Verwandlung auch die $(a_i; b_i)^3$ vollständig bestimmt, weil durch die einem ebenen Schnitte entsprechende Curve und durch $b_1^2, \ldots, b_4^2$ die $M_2^3$ vollständig bestimmt ist. Nach der anderen baut sie sich über einer in sich transformirten $M_1^4$ $(p = 0, 1)$ auf, deren (binäres) Gebiet in bekannter Weise in sich verwandelt wird. Dass im letzteren Falle die $(a_i; b_i)^3$ noch $\infty$ fach bestimmt ist, liegt in der Natur der Sache. Vorher finde folgendes brauchbare Theorem Platz.

*Theorem LXXV.— Wenn eine Transformation $(a_i;\ b_i)^3$ die Punkte $p$, $q$ in $p'$, $q'$ verwandelt, existirt eine Collineation mit $a_i$ in $b_i$ $(i=1, \ldots, 4)$, $p$ in $q'$, $q$ in $p'$.*

Lässt man nämlich bei festen $q'$, $a_i$, $b_i$ nur $p'$ variiren, so variirt $q$ gemäss einer Collineation, weil jeder Geraden eine Gerade entspricht, was mit den directiven Ebenenbüscheln bewiesen wird; kommt $p'$ nach $q'$, so entspricht diesem $p$, also $q'$ in $p$, kommt $p'$ nach $p'$, so $q$ nach $q$.

*Theorem LXXVI.— Wenn eine Characteristik n. 3–15 existirt, enthält sie stets eine invariante $M_1^2$, welche nicht zerfällt.*

Denn durch die 7, 8, 9 Punkte geht mindestens eine $M_2^2$, welche wenn überhaupt, nur in zwei Ebenen durch $a_1$ zerfallen könnte; aber in der quadratischen Characteristik $(a_2,\ a_3,\ a_4;\ b_2,\ b_3,\ b_4)$ gibt es kein invariantes Geradenpaar.

*Theorem LXXVII.—Die durch stereographische Projection der $M_2^2$ entstehende ebene Transformation ist selbst der in $(a_i;\ b_i)^3$ enthaltenen $B(a_2,\ a_3,\ a_4;\ b_2,\ b_3,\ b_4)$ äquivalent.*

Nach XIII entsteht $(e_1 e_2')$, $(e_2 e_1')$, $(e_3 e_3')$, $B(e_4, e_5, e_6;\ e_4', e_5', e_6')$ und durch Transposition mit $(e_1 e_2 e_3)^2$ $E_i$ in $e_k e_3$ in $e_k' e_3'$ in $E_l$, $E_3$ in $e_2 e_1$ in $e_1' e_2'$ in $E_3$, $E_4$ in $e_4$ in $(e_1' e_2' e_3' e_5' e_6')^2$ in $E_5' E_6'$, $E_5$ in .... $E_6' E_4'$, $E_6$ in .... $E_4' E_5'$ mit $B$. Hiebei kann $i=1,\ 2$, $k=1,\ 2$ sein, $l$ wird entsprechend 1 oder 2. Es entsteht also entweder ein Paar Doppelpunkte oder ein involutorisches Paar.

Die fertige Construction der Typen n. 3–15 ist also: Man nehme einen der 10 constructibeln ebenen Typen $B$ (oder I–X, Cr. J., p. 108), in demselben ein Tripel Doppelpunkte $f_1 f_2 f_3$ oder ein involutorisches Paar $i_1 i_2$ und einen Doppelpunkt $f_3$, übertrage $B$ durch $(f_1 f_2 f_3)^2$ oder $(f_3 i_1 i_2)^2$ in eine Transformation 5. Ordnung $Q^5$ und construire durch $f_1 f_2$ oder $i_1 i_2$ eine $M_2^2$ und projicire aus einem der Punkte $O$, deren Berührungsebene durch $f_1 f_2$, resp. $i_1 i_2$ geht, die $Q^5$ auf die $M_2^2$ und bestimme nach XIII durch $M_2^2$ die $(a_i;\ b_i)^3$. Wenn $f_1$, $f_2$ unendlich nahe sind, wird $M_2^2$ ein Kegel. Am Ende dieser Construction muss resultiren, dass die Geraden durch den Punkt von $M_2^2$, dessen Projection $f_3$ ist, unter einander gemäss $B$ transformirt werden. Hierauf basirend kann auch so construirt werden: Durch 2 Doppelpunkte $f_1 f_2$ oder ein involutorisches Paar $i_1 i_2$ von $B$ ziehe man $M_2^2$ und projicire aus dem Schnittpunkte $O$ zweier Erzeugenden durch $f_1$, $f_2$ oder $i_1$, $i_2$ die $B$ auf $M_2^2$, dann bestimmt sie dort eine $(a_i;\ b_i)^3$ mit $O$ als $(a_1 b_1)$. Hat demnach $B$ 6, 7, 8 Punkte, so hat $(a_i;\ b_i)^3$ 7, 8, 9 Punkte.

*Lemma.* In einem Netze von $M_2^3$ bilden die durch einen Basispunkt gehenden Erzeugendenpaare eine involutorische Verwandtschaft der Ordnung 8 mit den Stralen nach den anderen 7 als 3 fachen Stralen* (also $XLIII_3$ im Ternären).

*Theorem LXXVIII.—Der Index des Flächennetzes für n.* 3, 5, 6 *ist der Index, mit welchem die involutorischen Paare der $XLIII_3$ über B und einem ihrer Doppelpunkte unter einander vertauscht werden, also für n.* 3: 3, 6, 6, 6, *n.* 5: 9, 9, 9, *n.* 6: 3.

Der 8. Basispunkt des $M_2^3$-Netzes ist ein Doppelpunkt $d$ des $R_3$ und $a_1 d$ schneidet $B$ in einem Doppelpunkte. Die Erzeugendenpaare durch $a_1$ werden durch $(a_i;\, b_i)^3$ unter einander transformirt. Nun habe ich für die $B_6$, $B_9$, $B_{12}$ die Netze $C_3$ vollständig angegeben,† wonach:

*Theorem LXXIX.—Fur n.* 3 *gibt es zwei wesentlich verschiedene Varietäten, mit einer invarianten* $M_1^4$, $u' + \varepsilon u \equiv \gamma$, *einer* $M_1^4\, u' + u \equiv \gamma$ *und einem Geradenquadrupel oder mit* $u' + u \equiv \gamma$, $u - \varepsilon u \equiv \gamma$ *und* $M_1^4$ *mit Spitze; für n.* 5 *eine Varietät mit* 2 *invarianten* $M_1^4$ *mit Spitze und* $M_1^4$ $u' - \varepsilon u \equiv \gamma$; *für n.* 6 *zwei Varietäten mit Geradenquadrupel,* $u' + iu \equiv \gamma$ *und* $M_1^4$ *mit Spitze oder mit* $u' + \varepsilon u \equiv \gamma$, $u' + iu \equiv \gamma$ *und* $M_1^4$ *mit Spitze.*

Diese Varietäten sind in keiner Weise unter einander transponirbar, eine Erscheinung, welche auch bei anderen monoidalen Transformationen begegnet.— In jedem der obigen Fälle ist die Kegelspitzencurve auf dem Kegel, welcher von $(a_1 b_1)$ die Hessesche Curve des ausgewählten $C_3$-Netzes projicirt.

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* Ebenso bilden im $R_r$ die von einem Basispunkte eines $\infty^{r-1}$ Systemes von $M_{r-1}^2$ ausgehenden Erzeugendenkegel derselben ein lineares System, welches einen $R_{r-2}$ in einem linearen $\infty^{r-1}$ Systeme von $M_{r-3}^2$ schneidet. Dasselbe enthält $\infty^{r-2}$ dass Systeme angehörige (d. h. von $M_{r-2}^2$ erfüllte) $M_{r-1}^2$. Eine weitere Verallgemeinerung ist möglich, indem man $M_{r-1}^2$-Systeme mit gemeinsamen $R_i$ und die durch sie gehenden $R_{i+1}$ betrachtet oder zu $M_{r-1}^n$-Systemen übergeht. Ich benütze dieses Verfahren seit längerer Zeit zur Entdeckung von linearen $\infty^{r-i}$-Systemen von $M_i$ im $R_r$.

† Preisschrift II, §§12, 19, 26. Ich habe hier die Stelle "Ad notam" aus meinem citirten Buche zu verschärfen. Dort habe ich einige Worte über Autonne's Noten über einige höchst unbedeutende Gruppen aus lauter quadratischen und cubischen ebenen Transformationen gesprochen, obzwar es ganz überflüssig war. Ich habe aber dort nicht vollständig deutlich hervortreten lassen, dass die Noten Autonne's zu der Typentabelle aus meiner Preisschrift—also zur Theorie der einzelnen periodischen Transformationen—*gar keinen Bezug* haben. *Kein einziger* von den 48 Typen meiner Preisschrift oder der Abh. in Cr. J. oder Acta Math. ist auch nur angedeutet—auch nicht im entferntesten—in sämmtlichen Noten Autonnes; ja *nicht einmal* irgend ein einem jener Typen äquivalente, nicht reducirte Transformation. Und aber jene einfachsten, ganz elementaren und, wie gesagt, nicht typischen quadratischen und cubischen Tr. welche überhaupt bei ihm vorkommen, haben in meiner Preisschrift u. zwar schon in jener Fassung, welche 31. März 1883 der Akademie zu Neapel vorlag, doch wol eine ganz andere Behandlung erfahren.

Wenn $B$ 7 Punkte hat, können die $f_1 f_2$ oder $i_1 i_2$ mit $B$ $\infty^1$ oder eine $C_3$ bestimmen, wonach $(a_i; b_i)^3$ im $R_3$ 8 associirte Punkte oder nicht haben wird. Im ersten Falle gilt für den Index des $M_2^2$-Netzes LXXVIII, im anderen für das $M_2^2$-Büschel, dass der Index gleich dem Index unter den Punktepaaren auf $C_3(B, f_1 f_2$ oder $i_1 i_2)$. Für $B'_{12}$ gehen $C_h$ und $C_1 + C_2$ durch $i_1 i_2$, $C_h$ und $C_e$ $u' - \varepsilon u \equiv \gamma$ durch $d_2 d_3$, $C_1 + C_2$ und $C_e$ durch $d_1$ mit Contact* durch $d_1$, $d_3$ nur $C_e$, es gibt 4 Varietäten. Für $B_{14}$ gibt es eine Varietät mit $M_2^2$-Netz und drei invarianten Kegelflächen.† Für $B_{18}$ gehen $C_1 + C_2$ und $C_e(u' + \varepsilon u \equiv \gamma)$ durch $i_1 i_2$, $C_e$ und $C_3^3$ durch $d_1$ mit Contact, $C_3^3$ und $C_1 + C_2$ durch $d_2$ mit Contact,‡ $C_3^3$ allein durch $d_1 d_2$, daher gibt es zwei Varietäten. In jedem Falle ist der Index im Netze die Hälfte des Indexes der Transformation (wegen des Lemma). Daher:

*Theorem LXXX.—Für n. 7 sind entweder $M_1^4$ $u' + iu \equiv \gamma$, $M_1^4$ $u' - \varepsilon u \equiv \gamma$, $M_1^3$ mit Geraden (in involutorischer Vertauschung) invariant oder eine einzige $M_1^4$ $u' - \varepsilon u \equiv \gamma$; für n. 8 drei $M_1^4$ mit Spitze oder nur eine solche; für n. 9 entweder $M_1^4$ $u' + \varepsilon u \equiv \gamma$, $M_1^3$ mit Gerade, $M_1^4$ mit Spitze oder $M_1^4$ mit Spitze allein.*

Bei den Varietäten mit Büschel bleibt jedesmal ein Kegel und bei n. 8, 9 noch ein Kegel, bei n. 7 ein Hyperboloid invariant.

Wenn $B$ 8 Punkte hat, so gestatten $B_{18}$, $B_{20}$, $B_{24}$, $B_{30}$ je $f_1 f_2$, welche mit $B$ in einer $C_3^3$ sind, und $B_{30}$ $i_1 i_2$ mit $B$ in $C_h$, $B_{15}$, $B_{30}$ $i_1 i_2$ mit $B$ in $C_e$ $u' + \varepsilon u \equiv \gamma$, $B_{15}$, $B_{20}$, $B_{24}$ gestatten $f_1 f_2$ ohne solche Lage und getrennt, während $B_{30}$ sowie die übrigen drei zwei unendlich nahe $f_1 f_2$ gestatten, welche nicht in $C_3$ durch $B$ sind. Jeder der ersteren Lagen entspricht eine $(a_i; b_i)^3$ mit $M_2^2$-Büschel, jeder der letzteren eine $(a_i; b_i)^3$ mit einziger $M_2^2$ und diese ist, wenn $f_1$ unendlich nahe $f_2$ ein Kegel. Daher:

*Theorem LXXXI.—Für n. 10 bleibt entweder $M_1^4$ $u' + iu \equiv \gamma$ oder $M_1^4$ mit Spitze oder nur eine $M_2^2$ invariant; für n. 11 entweder $M_1^4$ mit Spitze oder nur eine $M_2^2$; für n. 12, 13 entweder $M_1^4$ $u' + \varepsilon u \equiv \gamma$ oder $M_1^4$ mit Spitze oder nur eine $M_2^2$; diese $M_2^2$ kann jedesmal auch ein Kegel sein.*

Ueber den Index unter den $M_2^2$ des Büschels entscheidet LXV, für $M_1^4$ mit Spitze müssen die beiden Kegel invariant sein. In keinem diese Fälle sind die Varietäten unter einander transponirbar. Bei n. 3, 12 gibt es 4, bei n. 5, 7, 8, 10, 11 gibt es 3, bei n. 6, 9, 13 gibt es 2 Doppelgeraden $(a_1 b_1)$, wenn kein Doppelpunkt in die $C_3$-Basis tritt und die Indices in den Doppelgeraden bestimmen sich durch die $M_2^2$.

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* Ib. II, §20. † Ib. II, §21. ‡ Ib. II, §27.

Die vorhergehenden Theoreme können auch durch die Parameter (§8) abgeleitet werden, womit die 2. Constructionsmethode durchgeführt wird. Ich begnüge mich mit den folgenden zwei allgemeinen Theoremen.

Sei $u' + xu = c$ die Projectivität in einer $M_1^4$ mit Spitze; ich nehme aber als Schnittpunkttheorem nicht $\sum u = 0$ sondern $\sum u = a_1$ an, was eine lineare Aenderung des Parameters ist. Dann werden die Formeln gelten: $b_i + xa_i = -c + a_1(1+x):2$, woraus $b_1 = -\frac{c}{1+x} + \frac{b_1}{2}$, $b_1 = -\frac{2c}{1+x}$ und die Formeln $a_i + xb_k = c$ gemäss $B$. Nun sind, wenn $M_1^4$ und die Projection aus $a_1$ derart bezogen werden, dass zwei mit $a_1$ alineirte Punkte gleichen Parameter haben, für $C_3^3$ die Projectivität $u' + xu = c$ und das Schnittpunkttheorem ist, weil 3 alineirte Punkte dieselben Parameter wie 3 mit $a_1$ complane Punkte der $M_1^4$ haben, $\sum^3 u = 0$. Also werden* die Formeln gelten $b_i + xa_i = -2c$, $a_i + xb_k = c$, also wegen des $b_1$-Werthes, dieselben wie im Raume. Berechnet man nun in der Ebene die Parameter $b_2$, $b_3$, $b_4$, so resultirt aus $b_2 + b_3 + b_4 = -3c$ eine Gleichung in $x$, $f(x) = -3$. Im $R_3$ werden nun die Parameter $b_2$, $b_3$, $b_4$ dieselben und es resultirt aus §8, wenn dasselbst statt des Schnittpunkttheorems das gegenwärtige $\sum u = k$ gesetzt wird, $\sum b = k - 2c + k(1+x):2$ und da $k = a_1$, $\sum b = k - 3c$, also $f(x) = -3$.

*Theorem LXXXII.—Auf einer Curve $M_1^4$ mit Spitze, für welche $\sum u \equiv a_1$ das Schnittpunkttheorem, hat man also nur die für eine ebene $C_3^3$ in Preisschrift II, §34 berechneten Parameter von $b_2$, $b_3$, $b_4$† mit $(a_1 b_1) = -2c:(1+x)$ zu verbinden, um die Characteristik von $(a_i;\ b_i)^3$ construirt zu haben.*

Sei für $p = 1$, $u' + xu \equiv \gamma$ die Correspondenz; das Schnittpunkttheorem sei $\sum u \equiv a_1$, was durch eine Aenderung der unteren Integralgränze erreichbar ist. Dann werden gelten $b_i + xa_i \equiv -\gamma + a_1(1+x):2 + \omega$, woraus $b_1 = -\gamma:(1+x) + b_1:2 + \omega:(1+x) + C:(1+x)$ oder $b_1 \equiv -2\gamma:(1+x) + (C + 2\omega):(1+x)$ und $a_i + xb_k \equiv C$ gemäss $B$. Man kann auch hier $M_1^4$ und die Projections — $C_3$ derart beziehen, dass zwei mit $a_1$ alineirte Punkte gleichen Parameter haben, sodass für $C_3\,\Gamma$ auch $u' + xu \equiv \gamma$, das Schnittpunktetheorem aus obigen Gründen

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* Preisschrift II, §5.

† Preisschrift II, §34.

$\sum_1^3 u \equiv 0$ wird. Also werden* die Formeln gelten $b_i + xa_i \equiv -2\gamma$, $a_i + xb_k \equiv \gamma$, also wegen des $(a_1 b_1)$ Werthes von den räumlichen Formeln auch nicht in Hinsicht auf des $\omega$ verschieden. Mit dem Schnittpunkttheorem $\sum u \equiv k$ wird aber $\sum b = k - 2\gamma + k(1+x):2+\omega$ und da $k \equiv a_1$ wird $\sum_1^4 b \equiv a_1 - 3\gamma$, oder $\sum_2^4 b \equiv -3\gamma$. Daher:

*Theorem LXXXIII.—Auf einer elliptischen Curve, für welche $\sum u \equiv a_1$ das Schnittpunktetheorem, hat man also nur die für eine ebene $C_3$ in Preisschrift II,* §12, 19–23, 26–28 *berechneten Parameter von $b_2$, $b_3$, $b_4$*† *mit $(a_1 b_1) \equiv -2\gamma:(1+x) + 2\omega:(1+x)$ zu verbinden, um die Characteristik von $(a_i;\, b_i)^3$ construirt zu haben.*

§11.—*Die typischen Transformationen mit $(a_1 b_1)$, $(a_2 b_2)$ oder $(a_1 b_2)$, $(a_2 b_1)$.*

*LXXXIV.—Jede Transformation $(a_1 b_1)$, $(a_2 b_2)$, $b_3$ in .... $b_3^h = a_4$, $b_4$ in .... $b_4^h = a_3$,*† *ist constructibel mit einer als invariant vorausgesetzten $M_2^2$.*

Denn, indem man die erste Methode des vorigen § anwendet, kann man immer zwei Doppelpunkte $f_1 f_2$ in der Ebene finden, welche mit $(a_2 b_2)$ (resp. nach früherer Bezeichnung mit $(aa')$) nicht alineirt sind. Construirt man über ihnen die $M_2^2$, so erhält man die Transformation.

*Corollar.* Wenn $h = 1$, gibt es noch eine 2. Varietät. Man kann dann ein mit $(aa')$ nicht alineirtes Paar $i_1 i_2$ in der Ebene finden, über diesem die Erzeugenden der $M_2^2$ errichten und wie früher construiren.

*Theorem LXXXV.—Jede Transformation aus LXXXIV ist auch mit einer als invariant vorausgesetzten $M_1^4$ mit Spitze construirbar und für die Parameter gilt LXXXI.*

Ich habe Preisschrift IV §7 bewiesen, dass überhaupt jede Jonquièressche Transformation $(ab)$ mit invarianter $C_3^2$ construirbar ist und hienach ist für die Ordnung 2 das cit. Theorem anwendbar.

*Theorem LXXXVI.—Die Transformationen $(a_1 b_1)$, $(a_2 b_2)$, $(a_3 b_4)$, $b_3$ in .... $b_3^h = a_4$ sind nur mit einer invarianten $M_2^2$ construirbar, welche ein Kegel ist, wenn $h > 1$.*

Denn in der ebenen Transformation $(aa')$, $(bc')$, $b'$ in .... $b'^{(h)} = c$ kann man für $h > 1$ kein involutorisches Paar und kein Paar $f_1 f_2$ finden ohne Alinea-

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* Preisschrift II, §4. † Damit n. 1 des §7 constructibel sei, muss $m_1 = m_2$ sein, oder $m_1 = 0$.

tion mit $(aa')$, weil $ab$ schon ein Doppelstral ist. Dagegen wird es auf dem 2. Doppelstrale einen Doppelpunkt geben, der mit seinem seitlichen unendlich nahen Doppelpunkte als $f_1 f_2$ genommen werden kann.

Für $h = 1$ gibt es eine Varietät mit Hyperboloid, wo die Erzeugenden durch $(a_1 b_1)$ involutorisch vertauscht werden. Für die Varietät mit $M_1^4$ gilt LXXXV auch hier.

*Theorem LXXXVII.—Die allgemeinste Form der Transformationen* $(a_1b_1)(a_2b_2)$ *kann construirt werden mittelst einer invarianten* $M_2^n$ $(a_1^{n-1}a_2^{n-1})$.

Denn $x_1 = n-1$, $x_2 = n-1$, $x_3 = 1$, $x_4 = 1$ ist anallagmatisch und in dem linearen Systeme aller solcher $M_2^n$, welche $a_1^{n-1}$, $a_2^{n-1}$ und alle Punkte der Characteristik enthalten, wird es stets eine invariante $M_2^n$ geben, welche nicht zerfällt. Dann wird wie für $M_2^3$ construirt. Eine Gerade der Ebene wird von $a_1$ aus in eine $M_1^n a_1^{n-1}$ projicirt, diese in eine $M_1^{n+2} a_1^n b_2 b_3 b_4$ verwandelt und diese in eine $M_1^2 (e_2 e_3 e_4)$, wo $e_2$, $e_3$, $e_4$ Projectionen von $b_2$, $b_3$, $b_4$ sind, herabprojicirt. Die $Q^2$ hat genau den Rest von $(a_i;\, b_i)^3$ zur Characteristik. Also: Man construire in der Ebene $(aa')$, $(bc')$, $b'$ in .... $b'^{(h)} = c$ und nehme in einer Geraden über $(aa')$ zwei Punkte $a_1 a_2$ als $(n-1)$ fache Punkte einer $M_2^n$ und sorge dafür, dass die Geraden derselben, welche von $(a_1 b_1)$ ausgehen, die Ebene in einem oder mehreren Cyclen der $Q^2$ schneiden. Dann liefert die Projection der $Q^2$ von $a_1$ aus auf die $M_2^n$ eine Verwandlung, welche die $(a_i;\, b_i)^3$ des Raumes vollständig bestimmt.

*Theorem LXXXVIII.—Die Transformation* $(a_1 b_2)$, $(a_2 b_1)$, $b_3$ *in* .... $b_3^h = a_4$, $b_4$ *in* .... $b_4^h = a_3$, *sowie* $(a_1 b_2)$, $(a_2 b_1)$, $(a_3 b_4)$, $b_3$ *in* .... $b_3^h = a_4$ *existirt stets mit einer als invariant vorausgesetzten* $M_2^2$, *welche im* 2. *Falle eine Kegelfläche sein muss.*

Die 1. Methode des §10 liefert in der Ebene $(e_1 e_2')(e_2 e_1')(e_3 e_4')(e_4 e_3') \ldots)^5$ und diese durch $(e_1 e_2 e_3)^2 E_3$ in $E_3$, $E_1$ in $E_3' E_2'$, $E_2$ in $E_3' E_1'$, $E_4$ in $E_1' E_2' E_3' E_5' E_6'$, $E_5$ in $E_3' E_6'$, $E_6$ in $E_3' E_5'$ also $(E_1 E_2'$, $E_2 E_1'$, $E_5 E_6'$, $E_6 E_5'$, $E_4^2 E_3'^{\prime})^3$ mit $E_6'$ in .... $E_5$, $E_5'$ in .... $E_6$. Wenn also $h > 1$, so muss $E_2' = E_1$, $E_1' = E_2$ sein. Die Construction ist vollendet mit jener einer ebenen cubischen Transformation $(ab)$, $(a_1 b_1)$, $(a_2 b_2)$, $b_3$ in .... $b_3'^{(h)} = a_3$, $b_4$ in .... $b_4'^{(h)} = a_4$, ihrer Transposition und der Errichtung einer $M_2^2$. Für die 2. Transformation erhält man $(E_5 E_6')$, also falls nicht $E_1$, $E_2$ unendlich nahe sind, nothwendig auch $(E_6 E_5')$. $E_1 E_2$ in unendlicher Nähe bedeutet aber die Kegelfläche.

Für $h = 1$ ist eine 2. Varietät mit $E_1' = E_1$, $E_2' = E_2$ zulässig, wodurch man Aequivalenz mit $(E_4 E_3'$, $E_6' E_6'$, $E_6 E_2')^2 E_2'$ in $E_5'$ in $E_6$, $E_2$ in $E_5$ in $E_3$ erhält.

Diese Transformation ist leicht construirbar* und die zweimalige Rücktransposition führt zu jener, über welcher die $M_2^2$ zu errichten ist.

*Theorem LXXXIX.—Die allgemeinste Form von* $(a_1 b_2)$, $(a_2 b_1)$ *kann mittelst einer als invariant vorausgesetzten* $M_2^n (a_1^{n-1} a_2^{n-1})$ *construirt werden.*

Denn wie in LXXXVII kann man stets eine solche $M_2^n$ für hinreichend grosses $n$ finden. Eine Gerade der Ebene wird dann von $a_1$ aus in eine $M_1^n (a_1^{n-1})$ projicirt, diese in eine $M_1^{n+2} (a_2^n b_2 b_3 b_4)$ verwandelt und diese von $a_1$ aus in eine $M_1^{n+1} (e_1^n)$ herabprojicirt, welche ausser durch $e_3'$, $e_4'$ (Projectionen von $b_3$, $b_4$) auch durch weitere $2n - 2$ Punkte einfach geht. Diese sind die Schnittpunkte der Ebene mit den durch $a_1$ gehenden einfachen Geraden der $M_2^n$. Denn $M_1^n$ schneidet die $2n - 2$ Geraden durch $a_2$, welche sich in die $2n - 2$ Geraden durch $a_1$ verwandeln und daher für die Schnittpunkte dieser mit $M_1^{n+1}$ feste Projectionen liefern. Es entsteht eine Jonquières'sche Transformation mit $(ab)$ und $2n - 2$ Coincidenzen einfacher Fundamentalpunkte (mit unbestimmter Directrixsubstitution) und zwei Verkettungen $e_4'$ in .... $e_3$, $e_3'$ in .... $e_4$, wo $(e_3 e_4')(e_4 e_3')$ überdies gepaart sind. Wegen der Projectivität unter den Ebenen durch $a_1 a_2$ folgt, dass die Directrixsubstitution für die Coincidenzen aus Cyclen der Ordnung $h + 1$ und etwa 1 oder 2 $(e_i e_i')$ bestehen muss. Hieraus folgt, dass $2n - 2$ mod. $(h + 1) \equiv 0, 1, 2$ sein muss. Nimmt man nun umgekehrt eine solche Transformation in $E_1$, errichtet über $(ab)$ eine Gerade für die beiden Punkte $a_1 a_2$ und construirt $M_1^n (a_1^{n-1} a_2^{n-1})$ mit den Geraden aus $a_1$ über die $2n - 2$ coincidirenden Fundamentalpunkte, so liefern die Projectionen $e_3 e_4$, $e_3' e_4'$ die weiteren Elemente für die Raumtransformation.

Nur für $n = 2$ und die Voraussetzung von zwei Doppelebenen als Tangentialebenen der $M_2^2$ durch $a_1 a_2$ kann $h + 1$ jeden Werth haben.

*Theorem LXXXX.*—$(a_1 b_2)$, $(a_2 b_1)$ *ist in* $M_1^4$ *mit Spitze nicht construirbar, in* $M_1^4$ *mit* $u' - u \equiv \gamma$, *nur wenn* $h_1 = h_2 = 1$, *in* $u' + u \equiv \gamma$ *nur für* $h = 1$, *in* $u' + iu \equiv \gamma$ *für* $h_1 = 0, 1$, $h_2 = 1$, *in* $u' - \varepsilon u \equiv \gamma$ *unconstruirbar, in* $u' + \varepsilon u \equiv \gamma$ *für* $h_1 = h_2 = 1, 5$.

In $M_1^4$ folgt aus $b_1 + x b_2 = -c$, $b_2 + x b_1 = -c$, $b_1 = b_2 \,|$ für $u' - \varepsilon u \equiv \gamma$, aus $b_1 - \varepsilon b_2 \equiv -\gamma + \omega$, $b_2 - \varepsilon b_1 \equiv -\gamma + \omega$, $b_1 \equiv b_2 \,|$ für $u' - u \equiv \gamma$ bei $h_1 = h_2 = 1$, $2\gamma \equiv 0$, $b_1 - b_2 \equiv -\gamma + \omega$, $a_3$ willkürlich, $b_3 - a_3 \equiv -\gamma + \omega$, $b_4 - a_3 \equiv -\gamma$, $a_4 - a_3 \equiv \omega$, $2b_1 + 2a_3 \equiv -\gamma + \omega \,|$ für $u' + iu \equiv \gamma$ bei $h_1 = h_2 = 1$ wird $b_1 - b_2 \equiv \frac{k_1 - k_2}{2}$,

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* Preisschrift II, §82.

$b_2(1+i) \equiv -\gamma+\omega+(k_1-k_2):2$, $b_3+b_4 \equiv -\gamma(1+i)+\omega$, $b_3 \equiv -ia_3-\gamma+\omega$, $a_4 \equiv \gamma(1+i)-a_3-i\omega$, $b_4 \equiv ia_3-i\gamma$, $a_3$ willkürlich | bei $h_1=h_2=2$ wird $4i\gamma \equiv 0$, $2b_1 \equiv 2b_2 \equiv 2b_3 \equiv 2b_4 \equiv 2a_3 \equiv 2a_4 \equiv -\gamma(1-i)+\omega(1-i)$, sodass zwei von den 6 Punkten zweimal zusammenfallen müssten* | bei $h_1=0$, $h_2=1$ der gleiche Grund | fur $u'+\varepsilon u \equiv \gamma$ bei $h_1=h_2=1$ aus $b_3+\varepsilon a_3 \equiv b_4+\varepsilon a_4 \equiv -\gamma+\omega$, $a_4+\varepsilon b_3 \equiv a_3+\varepsilon b_4 \equiv \gamma$, $b_3 \equiv b_4$ | ebenso $h_1=0$, $h_2=1$ | bei $h_1=h_2=2$, $b_1 \equiv b_2 \equiv (1-\varepsilon)\gamma:(\varepsilon^2-1)+\omega(1-\varepsilon):(1-\varepsilon^2)$, $b_3+b_4 \equiv 2\varepsilon^2\gamma+\omega$, $a_3+a_4 \equiv 2\gamma+\varepsilon^2\omega$ | bei $h_1=0$, $h_2=2$, $b_3(1-\varepsilon^2) \equiv \gamma(1-\varepsilon)$, $b_4(1-\varepsilon^2) \equiv 3\gamma\varepsilon^2$ | bei $h_1=h_2=3$ erscheinen drei Werthe für 6 Punkte, ebenso $h_1=0$, $h_2=3$, ebenso $h_1=h_2=4$ und $h_1=0$, $h_2=4$ | bei $h_1=h_2=5$, $b_3-b_4 \equiv -3\gamma+\varepsilon^2\gamma+\omega$, $4\gamma-2\varepsilon^2\gamma \equiv 0$.

Die $(h+1)$. Wiederholung von $(a_1b_1)$, $(a_2b_2)$, $b_4$ in .... $a_3$, $b_3$ in .... $a_4$ besitzt eine Fläche von Doppelpunkten der Ordnung $2(h+2)a_1^{2h+3}a_2^{2h+3}$, welche also die Hemicykelfläche von $(a_i;\, b_i)^3$ ist. Ebenso besitzt $(a_1b_1)(a_2b_2)(a_3b_4)b_4$ in .... $a_3$ eine Hemicykelfläche der Ordnung $h+3$ oder $h+4$, je nachdem $h$ ungerade oder gerade. $(a_1b_2)$, $(a_2b_1)$ besitzt eine Hemicykelfläche nur, wenn $h+1$ ungerade, diese ist dann $2(h+2)$. Ordnung.

*Theorem LXXXXI.—Drei allgemeine Paare einer involutorischen Collineation (centralen oder geschaarten) können stets als $a_1a_2$, $a_4b_4$, $a_3b_3$ einer Transformation $(a_1b_2)$, $(a_2b_1)$, $b_4$ in $a_3$, $b_3$ in $a_4$ genommen werden.*

Es ist nur LXXV auf die Punktepaare $b_4a_3$, $b_3a_4$ anzuwenden. Ebenso:

*Theorem LXXXXII.—Ein allgemeiner Cyclus und zwei Doppelpunkte einer Collineation Indexes 4 können stets als $(a_3b_3)(a_4b_4)$ und $a_1a_2$ einer $b_3$ in $a_3$, $b_4$ in $a_4$, $(a_1b_1)$, $(a_2b_2)$ genommen werden. Ueber derselben Figur existirt auch $(a_1b_2)$, $(a_2b_3)$, $(a_4b_4)$, $b_1$ in $b_1'$ in $a_3$.*

Man wendet LXXV auf $b_3$ in $a_3$, $b_4$ in $a_4$ oder $b_1$ in $b_1'$, $b_1'$ in $a_3$ an.

*Theorem LXXXXIII.—Ein involutorisches Paar, ein cyklisches Tripel und ein Doppelpunkt können stets als $a_1a_2$, $a_3b_3a_4$ und $b_3'$ von $(a_1b_2)$, $(a_2b_1)$, $(a_3b_4)$, $b_3$ in $b_3'$ in $a_4$ genommen werden. Ueber dieser Figur existirt auch $b_4$ in $b_4'$ in $a_4$, $(a_1b_2)$, $(a_2b_3)$, $(a_3b_1)$.*

Man wendet LXXV auf $b_3$ in $b_3'$, $b_3'$ in $a_4$ oder $b_4$ in $b_4'$, $b_4'$ in $a_4$ an.

## §12.—*Die Methode der invarianten $M_2^3$. Zwei Typen sind unconstruirbar.*

Indem ich die Characteristik $(a_ib_{i+1})$, welche hier im $R_3$ typisch ist, auf den letzten § verschiebe, wende ich mich zu n. 17–22 des §7.

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* Ich gebe der Kürze wegen nur die Resultate der Parameterrechnung.

*Theorem LXXXXIV.—Die Typen n. 17–22 besitzen, wenn sie existiren, nothwendig eine invariante $M_2^2$.*

Denn ein $\infty^4$, $\infty^3$, $\infty^2$ System von $M_2^2$ ist in sich transformirt und der Periodicität wegen gibt es wenigstens 5, 4, 3 getrennte invariante $M_2^2$. Aber es lässt sich aus den 5, 6, 7 Punkten höchstens ein invariantes Ebenenpaar zusammensetzen (u. zw. für den Index 6 oder 18).

*Theorem LXXXXV.—Die stereographische Projection der $M_2^2$ liefert Transformationen äquivalent bezüglich mit Collineation und $\Gamma_6$, §25* und $(\Gamma_{10})$, $B_{18}$ und $(\Gamma_{18})$, §11† und $(\Gamma_8)$, $B_{14}$ und $(\Gamma_{14})$, $(B_{30})$ und $(\Gamma_{30})$.*

n. 17.—$(e_1 e_i')$, $(e_2 e_k')$, $(e_3 e_4')$, $(e_4 e_5')$, $(e_5 e_6')$, $e_3'$ in $e_6$ gibt durch $(e_1 e_2 e_3)^2$ $E_1 E_2'$, $E_2 E_1'$, $E_5 E_6'$, $E_6 E_5'$, $E_4 E_3')^3$, $E_3$ in $E_3$, $E_3'$ in $E_6$ und durch $(E_4 E_3' E_1)^2$, falls $i = 1$, $k = 2$, $(E_1 E_5$, $E_4 E_3'$, $E_5 E_6)^3$, $E_3'$ in $E_2$ in $E_4$, $E_6$ in $E_1$, $E_3$ in $E_3$ und durch $(E_1 E_5 E_6)^2$ $E_3$ in $E_3$, $E_5$ in $E_6$ in $E_3'$ in $E_2$ in $E_4$ in $E_1$ in $E_5$, falls aber $E_1 = E_2'$ $(i = 2, k = 1)$, ist der Typus $\Gamma_6$ vorhanden.

n. 18.—Für $i = 1$, $k = 2$ resultirt ebenso $E_1 E_5$, $E_4 E_3'$, $E_5 E_3'^{(1)}$ mit $E_3'$ in $E_2$ in $E_4$, $E_3'^{(1)}$ in $E_6$ in $E_1$ reductibel durch $(E_5 E_3'^{(1)} E_6)^2$ auf $E_1 E_1$, $E_3'^{(1)} E_3'$, $E_5 E_3'^{(1)}$ mit $E_3'$ in $E_2$ in $E_4$ in $E_6$ in $E_5$ d. i. l. c. §25. $m = 4$. Für $i = 2$, $k = 1$ $(\Gamma_{10})$.

n. 19.—Für $i = 1$, $k = 2$ resultirt ebenso $E_1 E_5$, $E_4 E_3'$, $E_5 E_3'^{(1)}$, $E_3'$ in $E_2$ in $E_4$, $E_3'^{(1)}$ in $E_3'^{(2)}$ in $E_6$ in $E_1$, durch $(E_5 E_3'^{(1)} E_3'^{(2)})^2$ in $E_5 E_3'^{(1)}$, $E_1 E_6$, $E_3'^{(1)} E_3'$, $E_6$ in $E_1$, $E_3'$ in $E_2$ in $E_4$ in $E_3'^{(2)}$ in $E_1$ durch $(E_6 E_1 E_3'^{(1)})$ in $E_1 E_6$, $E_6 E_3'$, $E_5 E_1$, $E_3'$ in $E_2$ in $E_4$ in $E_3'^{(2)}$ in $E_5$, $E_3'^{(1)}$ in $E_3'^{(1)}$, also $B_{18}$. Für $i = 2$, $k = 1$ $(\Gamma_{18})$.

n. 20.—Für $i = 1$, $k = 2$ $E_1 E_6'$, $E_4 E_3'$, $E_5 E_6$, $E_3'$ in $E_2$ in $E_4$, $E_6$ in $E_1$, $E_6'$ in $E_5$,‡ was unconstruirbar. Für $i = 2$, $k = 1$ $(\Gamma_8)$.

n. 21.—Für $i = 1$, $k = 2$ $E_1 E_6'$, $E_4 E_3'$, $E_5 E_3'^{(1)}$, $E_3'$ in $E_2$ in $E_4$, $E_6'$ in $E_5$, $E_3'^{(1}$ in $E_6$ in $E_1$, äquivalent $B_{14}$. Für $i = 2$, $k = 1$ $(\Gamma_{14})$.

n. 22.—Für $i = 1$, $k = 2$ $(e_1 e_2 e_3)^2$ und $(E_4 E_3' E_1)^2 (E_4 E_1 E_3')^2 (E_2 E_3' E_1)^2$ geben $(B_{30})$. Für $i = 2$, $k = 1$ direct $(\Gamma_{30})$.

Wird von einem anderen als einem Doppelpunkte aus projicirt, so erhält man Transformationen derselben Classen, weil der Uebergang vom einen Projectionscentrum zum anderen in der Projection die Zusammensetzung mit einer $Q^2$ bedeutet. Damit dann aber die Transposition auf die genannten Typen im

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* Preisschrift II. † Ib.

‡ Ich bemerke, dass die quadratischen ebenen Characteristiken $a'$ in $b$, $b'$ in $a$, $c'$ in .... $c'^{(m)} = c$ durch $(a'bb')^2$ in $aa$, $b'b$, $c'a'$, $b'$ in $a'$, $c'$ in .... $c$ in $b$ übergeht, woraus für $m > 0$ die Unconstruirbarkeit ersichtlich ist.

wirklichen Falle möglich sei, werden allenfalls die Punkte $e$ besondere Lagen haben müssen.

*Theorem LXXXXVI.—Die Transformationen n.* 17–22 *haben, wenn sie existiren, denselben Index wie ihre Characteristik.*

Denn es können nicht 3 Punkte alineirt sein, weil durch die Wiederholung für 17. homaloidale Raumcurven 5. O. mit $p_1^2 p_2 p_3$ alineirt und $q^2$ ausserhalb der geraden Linie, für n. 18 und 20. homaloidale $M_1^7$ mit $p_1^2 p_2^2 p_3^2$ alineirt und $q^3$ ausserhalb, für n. 19 und 21, 22 homaloidale $M_1^{15}$ mit $p_1^4 p_2^4 p_3^4$, alineirt und $q^4$ ausserhalb resultiren würden, was die Characteristik ändern würde. Es können aus demselben Grunde nicht 4 Punkte in einer Ebene sein. Also hat die Collineation $T^6$, $T^{10}$, $T^{18}$, $T^8$, $T^{14}$, $T^{30}$ resp. 5, 6, 7, 6, 7, 7 unabhängige Doppelpunkte und ist die Identität.

*Theorem LXXXXVII.—Die Typen n.* 20 *und* 22 *sind nicht construirbar.*

1. Beweis. Denn die nach LXXXXV existirende $M_2^2$ führt nach LXXXXVI nothwendig auf eine unconstruirbare ebene Transformation, sei es dass man von einem Doppelpunkte oder einem gewöhnlichen Punkte projicirt.

2. Beweis. Die Anwendung von LXXV auf. n. 20 verlangt, dass die Raumcollineation vom Index 4 einen Cyclus und ein involutorisches Paar in allgemeiner Lage haben müsste, was nie der Fall ist. Für n. 22 ist der analoge Beweis etwas umständlicher.

3. Beweis. n. 20 Parameter in $M_1^4$ mit Spitze: $b_1 + xb_2 = b_2 + xb_3 = b_3 + xa_3 = b_4 + a_4 = -c$, $xb_4 + a_3 = xb_1 + a_4 = c$ liefern $b_4(1-x^2) = (-3 + x - 2x^2)c$, $b_3(1-x^2) = (-5 + x)c$, $b_2(1-x^2) = (-1 + 5x)c$, $b_1(1-x^2) = (-1 + x - 4x^2)c$, $\sum b(1-x^2) = -6x^2 + 8x - 10 = -2(1-x^2)$, $8x(1-x) = 0$, $x = 0, 1$.

In $u' - u \equiv \gamma$: $b_3 - a_3 \equiv -\gamma + \omega$, $b_4 - a_4 \equiv -\gamma + \omega$, $-b_4 + a_3 \equiv \gamma$, $-b_1 + a_4 \equiv \gamma$, woraus $b_1 \equiv b_3$. In $u' + u \equiv \gamma$: dieselben Schlüsse. In $u' + iu \equiv \gamma$, dieselben Schlüsse. $u' \pm \varepsilon u \equiv \gamma$ ist mit dem Index unverträglich.

n. 22.—In $M_1^4$ mit Spitze, $u' \pm u \equiv \gamma$, Schlüsse wie bei n. 20 auf $b_1 \equiv b_3$. In $u' + \varepsilon u \equiv \gamma$ aus denselben Formeln $b_1 \equiv b_3$.

## §13.—*Der Typus mit dem Index* 6.

1. *Theorem LXXXXVIII.—Ein allgemeiner Cyclus einer Raumcollineation vom Index 6 kann stets als $a_1 a_2 a_3 b_1 d a_4$ einer Transformation $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ in $a_4$ benützt werden, welche $d$ zum Doppelpunkte hat.**

Man wendet LXXV auf $b_2$ in $a_4$, $d$ in $d$ an.

*Theorem LXXXXIX.—Irgend eine Folge von 7 successiven Punkten in einer auch nicht periodischen Raumcollineation kann als $pa_4 a_3 a_2 a_1 b_1 p'$ einer Transformation $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ in $a_4$ benützt werden, welche $p$ in $p'$ überführt.*

Man wendet LXXV auf $b_1$ in $a_4$, $p$ in $p'$ an.

2. $T^2$ ist $(b_1 a_4, a_2 b_1, a_4 b_2, b_4 a_1)^3 a_1$ in $b_4$, hat also auf der Geraden $a_1 b_4$ stets 2 Doppelpunkte, $\delta_1 \delta_2$ die Hessesche Covariante von $a_1 b_4 \omega$, wo $\omega$ in der Ebene $b_1 b_2 a_4$ und wird durch $(\delta_1 a_2 a_4 b_1)^3$ in die Collineation $A_1$ in $B_4$ in $\Delta_1$ in $A_1$, $A_2$ in $B_1$ in $A_4$ in $A_2$, $\Delta_2$ in $\Delta_2$ übertragen. In der Ebene $A_2 B_1 A_4$ sind ausser $\Delta_1$ noch 2 Doppelpunkte $\Delta_3$, $\Delta_4$ enthalten, von welchen einer, $\Delta_3$ etwa, mit $\Delta_2$ verbunden eine Gerade von $\infty^1$ Doppelpunkten bestimmen muss, und welcher in $T^2$ eine $M_1^3$ durch $b_1 a_2 a_4 \delta_1 \delta_2 \delta_3$, wo $\delta_3$ unendlich nahe an $\delta_1$, entspricht. Folglich:

*$T$ besitzt $\infty^1$ involutorische Paare in einer $M_1^3$ $(T_3)$, welche $b_1 a_2 a_4$ enthält und $a_1 a_3$ in zwei Punkten $\delta_1 \delta_2$ trifft, $b_1 a_4$ und $\delta_1 \delta_2$ als involutorische Paare, $a_2$ und einen Punkt $d$ als Doppelpunkte enthält.* Diese binäre Vertheilung folgt aus den Eigenschaften der Covariante $Q$ der cubischen Form, welche $\Delta_2 \Delta_3$ auf den Ebenen $\Delta_1 (A_2 A_4, A_4 B_1, B_1 A_2)$ ausschneidet, $d$ ist auch Doppelpunkt von $T$. Die $M_1^3$ durch $b_1 a_2 a_4$, welche $a_1 a_3$ zweimal schneiden, werden alle unter einander transformirt. Die $M_2^2$ durch $b_1 a_2 a_4$ und die Gerade $a_1 a_3$ werden unter einander und ebenso die $M_2^2$ durch $I_3$ und die Gerade $a_1 a_3$ unter einander transformirt. Diese letzteren sind mit dem Index $\delta$ transformirt und die beiden Kegeln mit $\delta_1$, $\delta_2$ als Spitzen sind invariant. Die Doppelgerade $\Delta_2 \Delta_4$ gibt in $T^2$ eine $M_1^3 (X_3)$, welche ebenfalls $b_1 a_2 a_4 \delta_1 \delta_2$ enthält, invariant ist mit dem Index 3 und $\delta_1 \delta_2$ als Doppelpunkte, $b_1 a_2 a_4$ als periodisches Tripel hat. Die $M_2^2$ durch $a_1 a_3$ und $X_3$ schneiden $I^3$ in 1 Punkte, werden also mit dem Index 2 transformirt, sodass die Kegeln

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*** Nur der Priorität wegen mache ich auf die sich bietenden endlichen Gruppen cubischer Transformationen über 5 und höherer Transformationen über 6 Punkten aufmerksam, welche den Gruppen $M_4$, $M_5$ in II. Theil §5 meines Buches (Berlin, Mayer & Müller 1895) analog sind.—Diese Gruppen werden in meiner Abhandlung Acta Math. 1897 behandelt werden.**

mit den Spitzen $\delta_1$, $\delta_2$ ein involutorisches Paar bilden und invariant bleiben: eine Fläche $R_2$ durch $d$ und eine 2. Fläche $S_2$.

3. $T^3$ ist $(a_3 a_4, a_4 a_3, b_1 b_4, b_4 b_1)^3 a_2$ in $a_2$ und hat in Folge dessen (§19) $\infty^2$ Doppelpunkte in einer $M_2^2$ durch die Geraden $a_4 b_4$, $a_4 b_1$, $a_1 b_4$, $a_1 b_1$, welche auch $a_2$ enthält. Daher:

*Theorem C.—T besitzt $\infty^2$ periodische Tripel in einer $M_2^2$, welche $a_1 b_4$, $a_4 b_1$, $a_1 b_1$, $a_4 b_4$ ganz sowie die $X_2$ und d enthält und also $R_2$ ist. Die* 4 *durch d gehenden invarianten $M_2^2$ sind die* 3 *Kegeln mit den Spitzen in d, $\delta_1$, $\delta_2$ und $R_2$. Die Projectivitäten in den von $R_1$ mit den $\delta_1$, $\delta_2$-Kegeln geformten Büscheln sind vom Index* 6.

Je zwei Kegelflächen mit den Spitzen in einem involutorischen Paare auf $I^3$ bestimmen ein invariantes Büschel, dessen Basis nach §8. I n. nur $M_1^4$ mit $u' - u \equiv \gamma$ sein kann. Nun liefert die Parameterrechnung hiefür $b_1 - b_2 \equiv -\gamma + \omega$, $b_2 - b_3 \equiv -\gamma + \omega$, $b_3 - b_4 \equiv -\gamma + \omega$, $b_4 - a_4 \equiv -\gamma + \omega$, $a_4 - b_1 \equiv \gamma$, woraus $b_1 \equiv b_2 - \gamma + \omega$, $b_3 \equiv b_2 + \gamma + \omega$, $b_4 \equiv b_2 + 2\gamma$, $\sum b \equiv 4b_2 + 2\gamma \equiv -2\gamma + \omega$ (nach §8), $4b_2 \equiv -4\gamma + \omega$, ferner $b_1 - a_4 \equiv -4\gamma$, daher $-4\gamma \equiv -\gamma$, $3\gamma = 0$. Der Index ist also 3 und alle diese $\infty^1$ $M_1^4$ sind in $R_2$ und unter ihnen ist die Schnittcurve des $d$-Kegels und die Schnittcurve des Ebenenpaares $a_2 b_2 a_4 + a_2 b_1 a_1$ mit $R_2$ enthalten. Dieses Ebenenpaar ist thatsächlich als invariante $M_2^2$ anzusehen, indem die Ebenen vertauscht werden. Die Indices dieser Büschel sind sämmtlich 2. In jedem ist eine 2. invariante $M_2^2$ enthalten, $A_2$ und diese bilden ein Büschel mit dem Index 1, enthaltend $S_2$, das Ebenenpaar und den $d$-Kegel. Jede dieser $M_2^2$ schneidet sowohl den $\delta_1$- als den $\delta_2$-Kegel in einer invarianten $M_1^4$ $u' + \varepsilon u \equiv \gamma$.

Parameter in $u' + \varepsilon u \equiv \gamma$. $b_1 + \varepsilon b_2 \equiv b_2 + \varepsilon b_3 \equiv b_3 + \varepsilon b_4 \equiv b_4 + \varepsilon a_4 \equiv -\gamma + \omega$, $a_4 + \varepsilon b_1 \equiv \gamma$ liefern $b_1 + b_4 \equiv 2\varepsilon\gamma$, $\varepsilon^2 b_1 - b_4 \equiv -\varepsilon^2\gamma + \omega\varepsilon$, $b_4 \equiv \varepsilon^2\gamma - \omega$, $\sum b \equiv -2\gamma + \omega\varepsilon^3 \equiv -2\gamma + \omega$ (nach §8), $\omega \equiv 0$ und $b_3 \equiv -\varepsilon\gamma + \omega\varepsilon \equiv -\varepsilon\gamma$, sodass $b_3$ der Doppelpunkt von $u' + \varepsilon u \equiv \gamma$ wird, wie es sein muss. Ausserdem $b_1 \equiv (\varepsilon - 2)\gamma$, $b_2 \equiv (\varepsilon^3 - 1)\gamma$, $b_4 \equiv (1 - \varepsilon^3)\gamma$.

Der $d$-Kegel schneidet die $\delta_1$-, $\delta_2$-Kegel in $M_1^4$ mit Spitzen, wo die Parameterrechnung liefert: $b_1 + xb_2 = b_2 + xb_3 = b_3 + xb_4 = -c$, $a_4 + xb_1 = c$; $b_3 - x^3 a_4 = -c + cx$, $b_2 + x^3 a_4 = -c + cx - cx^2$, $b_1 - x^4 a_4 = -c + cx - cx^2 + cx^3$, $a_4(1 + x^5) = c(1 + x - x^2 + x^3 - x^4)$, $b_1(1 + x^5) = c(-1 + x - x^2 + x^3 + x^4)$, $b_2(1 + x^5) = c(-1 + x - x^2 - x^3 - x^4)$, $b_3(1 + x^5) = (-1 + x + x^2 + x^3 - x^4)$, $b_4(1 + x^5) = c(-1 - x - x^2 + x^3 - x^4)$, $\sum b(1 + x^5) = c(-4 + 2x - 2x^2$

$+ 2x^3 - 2x^4) = -2c$ (nach §8), also $(x^3 - 1)(x^2 - x + 1)$; $x = -\varepsilon$ liefert $b_3 = b_4$, $x = +\varepsilon$ gibt aber Index 6 und insbesondere $b_2 + b_4 = 0$, was ausdrückt, dass die Spitze des einfach zählenden Kegels auf $b_3 b_4$ enthalten ist, also entweder $\delta_1$ oder $\delta_2$ sein muss.

Für $M_1^4$ mit Doppelpunkt (Schnittcurve von $R_2$ mit $d$-Kegel) wird: $b_4 = \pm a_4 : x$, $b_3 = \pm b_4 : x$, $b_2 \equiv \pm b_3 : x$, $b_1 = \pm b_2 : x$, $a_4 = b_1 x$, woraus durch $b_1 b_2 b_3 b_4 = \mp k : x^3$, $a_4^4 = \mp k$ und $x^3 = 1$, also wirklich der Index 3 folgt. Zusammenfassend:

*Theorem CI.—Der Typus Index* 6 *besitzt* 1. *eine Tripelfläche $R_2$,* 2. *einen Kegel mit $d$,* 3. *einen mit $\delta_1$,* 4. *einen mit $\delta_2$,* 5. *ein Ebenenpaar als invariante $M_2^2$, sodass die Büschel aus* 2. *und* 5., 1. *und* 2., 1. *und* 5., 1. *und* 3., 1. *und* 4., 2. *oder* 5. *mit* 3. *oder* 4., 3 *und* 4 *resp. die Indices* 1, 2, 2, 6, 6, 3, 3, 3, 3, 3 *besitzen.*

## §14.—*Der Typus mit dem Index* 10.

1. *Theorem CII.—Einen allgemeinen Cyclus einer Raumcollineation vom Index* 5 *nebst einem Doppelpunkte kann man immer als $a_1 b_1 a_4 a_3 a_2$ und $b_1'$ einer Transformation $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $b_1$ in $b_1'$ in $a_4$ nehmen.*

Man wendet LXXV auf $b_1$ in $b_1'$ und $b_1''$ in $a_4$ an.

2. $T^2$ ist $(a_2 b_1, a_4 b_3, b_4 b_2, b_1' b_1')^3\, b_1$ in $a_4$, $b_2$ in $b_4$, und wird durch $(b_1' a_4 a_2 b_1)^3$ transponirt auf $A_4$ in $A_2$ in $B_1$ in $B_2$ in $B_4$ in $A_4$, $B_1'$ in $B_1'$. Da $T^5$ eben $(4, \ldots 4)^7$ ist, folgt:

*Theorem CIII.—Setzt man über einem Cyclus und einem Doppelpunkte einer Raumcollineation Index* 5 *eine typische involutorische Transformation* 7. *Ordnung fest und combinirt beide, so wird das Resultat durch angemessene Transposition in den cubischen Typus Index* 10 *verwandelt.**

Die Ebenen $a_1 a_3 b_1'$, $a_2 b_1 a_4$ werden vertauscht, das Ebenenpaar ist also eine invariante $M_2^2$, die Schnittgerade wird, wie $T^5$ lehrt, mit dem Index 10 transformirt und hat einen Doppelpunkt in $\tau = (a_1 a_3;\, a_2 b_1 a_4)$ und einen Doppelpunkt $d_1$. Die Geraden um $b_1'$ in $a_1 a_3 b_1'$ werden in die Kegelschnitte um $a_2 b_1 a_4$ transformirt und $b_1' d_1 + a_2 b_1 a_4 \tau d_1$ ist eine invariante zerfallende $M_1^3$ und mit $a_1 a_3$ eine invariante $M_1^4$.

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* Wird zur Transposition von $T^2$ die involutorische Transformation $(b_1' b_1', a_3 a_3, a_2 a_2, b_1 b_1)^3$ $b_2$ in $b_4$ in $b_2$ benützt, so wird $a_4$ in $a_2$ in $b_1$ in $b_4$ in $b_2$ in $a_4$, $b_1'$ in $b_1'$ das Resultat, dies ist die 2. Wiederholung der gemäss CI ohnehin unter den 6 Punkten bestehenden Collineation.

Die mit $T^2$ äquivalente Collineation hat eine invariante $M_1^3$ durch $B_1'$ und einen invarianten Kegel, dessen Spitze der 2. Doppelpunkt $D_1$ in dieser $M_1^3$ ist; diese übertragen sich in die Gerade $a_1 a_3$ und einen für $T$ invarianten Kegel, dessen Spitze in $\sigma$ auf $a_1 a_3$ sei, $D_1$ aber nach $d_1$. Die $M_1^3$ durch $a_3 b_1 a_4 b_1'$ mit $a_1 a_3$ als Bisekante werden unter einander und jene, welche $a_1 a_3$ in $\sigma$ berührt, wird in sich transformirt und enthält noch einen 2. Doppelpunkt $d_2$. *$d_1$ bestimmt ein Netz von $M_2^2$ mit Berührung in $d_1$, daher auch $d_2$ ein Netz mit Berührung in $d_2$. Die invarianten $M_2^2$ des* 1. *sind das Ebenenpaar, der $\sigma$-Kegel und eine Fläche $A_2$, des* 2. *der $d_2$-Kegel, der $\sigma$-Kegel und $A_2$.* Die beiden Kegel haben als Schnitt eine $M_1^4$ mit Spitze, der $d$-Kegel und $A_2$ eine $M_1^3$ mit Geraden $b_1' d_2$; das erstere wird auch durch die hier folgenden Parameter $b_2 + b_4 = 0$ bewiesen.

Parameter auf $M_1^4$ mit Spitze. $b_1 + xb_2 = b_2 + xb_3 = b_3 + xb_4 = b_4 + xa_4 = -c$, $d_4 + xb_1' = c$, $b_1' + xb_1 = c$ liefern $a_4(1 - x^6) = c(1 - x - x^2 + x^3 - x^4 + x^5)$, $b_1'(1 - x^6) = c(1 + x - x^2 + x^3 - x^4 - x^5)$, $b_1(1 - x^6) = c(-1 + x - x^2 + x^3 + x^4 - x^5)$, $a_1(1 - x^6) = c(-1 + x - x^2 - x^3 + x^4 + x^5)$, $a_2(1 - x^6) = c(-1 + x + x^2 - x^3 - x^4 + x^5)$, $a_3(1 - x^6) = c(-1 - x + x^2 + x^3 - x^4 + x^5)$, woraus $(x - 1)(x^5 - 1) = 0$, also Index 10 in $M_1^4$ und die Parameter folgen.

Parameter auf $M_1^4$ mit Doppelpunkt. $b_1 : b_2 = b_2 : b_3 \doteq b_3 : b_4 = b_4 : a_4 = \pm 1 : x$, $a_4 = xb_1'$, $b_1' = xb_1$ liefern $x^2 = 1$, was wegen $b_1$ in $b_1'$ in $a_4$ unmöglich ist.

Die stereographische Projection von $A_2$ aus $d_2$ liefert eine Transformation 4. Ordnung $Q_4$, welche die Projectionen von $a_1 a_3$ und $M_1^3$ vertauscht, jene aus $d_1$ ebenfalls $Q_4$, welche die Projectionen des Ebenenpaares vertauscht, und auch die Projectionen von $b_1' d_2$ und $M_1^3$ unter einander vertauscht. Analog die Projectionen des $\sigma$-Kegels.

## §15.—*Der Typus mit dem Index* 18.

1. Die Anwendung von LXXV lehrt, das $a_1 b_1 a_4 a_3 a_2$ einen Cyclus einer Raumcollineation bildet, in welcher $b_1''$ in $b_1'$ ein Paar ist, welches also nur noch passend gewählt werden muss. Andererseits besteht die Collineation (auch wegen LXXV) $a_4 a_3 a_2 a_1 b_1 b_1''$ mit $b_1'$ in $b_1'$. Lässt man also $b_1''$ variiren, so hat man jedesmal 1 Punkt $b_1'$ und 4 Punkte $b_1'$ im Entsprechen und die Doppelpunkte dieser Verwandtschaft (1, 4) sind die Punkte $b_1'$, welche den Typus liefern. Ich werde diese Verwandtschaft im $R_r$ genauer studiren.

2. $T^2$ ist $(b_1' b_1'', b_3 a_4, b_1 a_3, b_2 a_3)^3\, b_1$ in $b_1''$, $b_1'$ in $a_4$, $b_2$ in $b_4$ also nach Theorem XLIII äquivalent n. 5 des §7. Daher:

*Theorem CIV.—Der Typus Index* 18 *kann construirt werden, indem man über der Figur von* $(a_1 b_1)$, $(a_2 b_3)$, $b_4$ *in* $a_3$, $b_2$ *in* $b_2'$ *in* $a_4$ *den involutorischen Typus* 15. *Ordnung construirt und mit jener Transformation zusammensetzt und hernach eine angemessene Transposition anwendet.*

Man beachte, dass in Theorem LXXIX bewiesen worden, dass es von $(a_1 b_1) B_9$ nur eine Varietät gibt. Da ferner $Q_{15}$ die $M_2^2$ alle in sich transformirt, so ist die Verwandlung der $M_2^2$ im Typus dieselbe wie in $(a_1 b_1) B_9$; also, mit $d_3$ den zur Characteristik associirten Punkt bezeichnend:

*Theorem CV.—Der Typus mit dem Index* 18 *besitzt zwei invariante* $M_1^4$ *mit Spitze und eine* $M_1^4$ *mit* $u' + \varepsilon u \equiv \gamma$, *deren Doppelpunkt* — $\gamma\varepsilon$ *in* $d_1$ ist.

Die $M_1^3$, welche $a_3 b_1 b_1' b_1'' a_4$ enthält und $a_1 a_3$ als Bisekante hat, berührt $a_1 a_3$ in $\tau$, sodass $\tau$ die Spitze eines invarianten Kegels ist und enthält noch $d_1$. Die 2. invariante $M_2^2$ des Büschels $(M_1^3 + a_1 a_3)$ ist $A_2$ und sie schneidet den $d_2$-Kegel in einer $M_1^4$ $u' + \varepsilon u \equiv \gamma$, während $\delta_2$-Kegel und $\tau$-Kegel sich in $M_1^4$ mit Spitze schneiden.

Parameter auf $M_1^4$ mit Spitze. $b_4 + x a_4 = b_1 + x b_2 = b_2 + x b_3 = b_3 + x b_4 = -c$, $b_1' + x b_1 = b_1'' + x b_1' = a_4 + x b_1'' = c$ liefern $a_4 (1 + x^7) = c(1 - x + x^2 + x^3 - x^4 + x^5 - x^6)$, $b_1 (1 + x^7) = c(-1 + x - x^2 + x^3 + x^4 - x^5 + x^6)$, $b_2 (1 + x^7) = c(-1 + x - x^2 - x^3 + x^4 - x^5 - x^6)$, $b_3 (1 + x^7) = c(-1 + x + x^2 - x^3 + x^4 + x^5 - x^6)$, $b_4 (1 + x^7) = c(-1 - x + x^2 - x^3 - x^4 + x^5 - x^6)$, $\sum b (1 + x^7) = c(-4 + 2x - 2x^3 + 2x^4 + 2x^5 - 2x^6) = -2c(1 + x^7)$ (nach §8), also $x^7 - x^6 + x^4 - x^3 + x - 1 = (x - 1)(x^6 + x^3 + 1)$ und $b_2 + b_4 = 0$, was bedeutet, dass die einfach zählende Kegelspitze auf $b_2 b_4$ ist.

Parameter auf $u' + \varepsilon u \equiv \gamma$. Die analogen Congruenzen liefern $a_4 (1 + \varepsilon) \equiv -\gamma + \omega - 4\varepsilon\gamma$, woraus $6(\varepsilon - \varepsilon^2)\gamma \equiv \omega$ und $b_1 \equiv (4 + \varepsilon)\gamma - \varepsilon\omega$, $b_2 \equiv (-4\varepsilon^2 + \varepsilon)\gamma - \varepsilon\omega$, $b_3 \equiv 5\varepsilon\gamma - \varepsilon\omega$, $b_4 \equiv (\varepsilon - 4)\gamma - \omega\varepsilon$, $b_1' \equiv (2 - 3\varepsilon)\gamma + \varepsilon^2\omega$, $b_1'' \equiv (2\varepsilon^2 - 3\varepsilon)\gamma - \omega$.

## §16.—*Der Typus mit dem Index* 14.

1. Die Anwendung von LXXV liefert auch hier zwei Collineationen: $a_4$ in $b_4$ in $b_1'$ in $b_3$ in $b_2$ in $b_1$ in $a_3$, und $a_4$ in $b_4$ in $a_4$, $b_1'$ in $a_3$ in $b_3$ in $b_2$ in $b_1$. Die erste ist, der 4 Punkte nicht complan sind, vom Index 4, die 2. aperiodisch. Aus ihnen wird folgen $a_3$ in $b_3$ in $b_2$ in $b_1$ in $a_3$ in $b_4$, $b_1'$ in $b_1'$, welche sich namentlich zur Construction eignet.

2. $T^2$ ist $(a_2^2 b_2^2,\ a_3^2 b_1^2,\ a_4 b_1',\ b_1' a_2,\ a_1 a_3,\ b_4 b_4)^5 b_1$ in $a_4$. Die Characteristik ist äquimultipel nach meiner Definition in Cr. J., Bd. CXIV und besitzt 4 uneigentliche Doppelpunkte, ist daher zufolge einem für den $R_3$ analogen Theorem zu dem Theoreme l. c. äquivalent zum collinearen Cyclus. Also:

*Theorem CVI.—Der Typus Index* 18 *kann construirt werden, indem man über einem allgemeinen Cyclus einer Raumcollineation Index* 7 *einen involutorischen Typus der* 15. *Ordnung construirt, diesen mit der Collineation zusammensetzt und nachher eine angemessene Transposition anwendet.*

Von den 4 eigentlichen Doppelpunkten ist $d_1$ der 8. Basispunkt des invarianten $M_2^2$-Netzes, $d_2 d_3 d_4$ sind Spitzen invarianter $M_1^4$, und die $d_2$-, $d_3$-, $d_4$-Kegel gehen resp. auch durch $d_3$, $d_4$, $d_2$ und berühren daselbst den $d_3$-, $d_4$-, $d_2$-Kegel. Andere invariante $M_2^2$ als diese Kegel existiren nicht. Die Projectivitäten in den $M_1^4$ sind vom Index 14.

Parameter auf $M_1^4$ mit Spitze: $a_3(1+x^7) = c(1+x+x^2-x^3-x^4+x^5-x^6)$, $a_4(1+x^7) = c(1-x-x^2+x^3-x^4-x^5-x^6)$, $b_1(1+x^7) = c(-1+x-x^2-x^3-x^4-x^5+x^6)$, $b_2(1+x^7) = c(-1+x+x^2+x^3+x^4-x^5-x^6)$, $b_3(1+x^7) = c(-1-x-x^2-x^3+x^4+x^5-x^6)$, $b_4(1+x^7) = c(-1-x+x^2+x^3-x^4+x^5+x^6)$, $b_1'(1+x^7) = c(1-x-x^2+x^3-x^4-x^5-x^6)$.

## §17.—*Ableitung der 6-punktigen Typen aus der Kummer'schen Fläche.*

Herr Reye und nach ihm W. Stahl haben für die Untersuchung der Stralencongruenzen 2. O. und Cl. eine (1, 2) deutige Verwandtschaft untersucht, welche den Ebenen des Raumes $R_3^1$ die $M_2^2$ durch 6 feste Punkte $p_1 \ldots p_6$ in $R_3$, den Punkten einer hiebei entstehenden Kummer'schen Fläche $K_4$ in $R_3'$ die Punkte der Kernfläche $\Phi_4$ von $R_3$, den 16 Doppelpunkten von $K_4$ die 15 Geraden $p_i p_k$ und die $M_1^3 (p_1 \ldots p_6)$ und den 16 Doppelebenen die 6 Punkte $p_i$ und die 10 Ebenenpaare $(p_i p_k p_l,\ p_m p_n p_o)$ entsprechen macht.

Diese (1, 2) deutige Transformation bietet sich nun hier ganz von selbst dar. Wenn über 6 Punkten eine Characteristik construirt ist, sei es dass sie alle 6 Punkte absorbirt, sei es nur 4 oder 5 Punkte und die anderen 2 oder 1 zu involutorischen oder Doppelpunkten hat, so transformirt sie die $\infty^3$ $M_2^2$ durch $p_1 \ldots p_6$ unter einander, also auch die Punktepaare, in denen sie sich schneiden, und wird also durch die Reye'sche Abbildung in eine Collineation des $R_3'$ übertragen, welche, da in $R_3$ die $\Phi_4$ reproducirt war, $K_4$ in sich transformiren wird.

Da durch die eindeutige Correspondenz in $\Phi_4$ die Raumtransformation bestimmt ist:

*Theorem CVII.—Die Reciprokaltransformationen sowie die von ihnen gebildeten endlichen Gruppen über 6 Punkten können durch 2-deutige Abbildung aus den Collineationen und Collineationsgruppen gewonnen werden, welche eine Kummer'sche Fläche in sich transformiren.*

Den Punkten $p_1, \ldots . p_6$ entsprechen 6 Ebenen $P_1, \ldots . P_6$ durch einen Doppelpunkt und das von ihnen an ihrem Kegel gebildete Sextupel ist projectiv dem von $p_1, \ldots . p_6$ auf $M_1^3(p_1, \ldots . p_6)$ gebildeten Sextupel. Da nun in der hier folgenden Note bewiesen wird, dass die Gruppen an $K_4$ nur von diesem Sextupel abhängen, so folgt, was auch schon aus dem Theorem LXXV einzeln gefolgert wurde:

*Theorem CVIII.—Die Existenz einer Gruppe aus $(a_i; b_i)^3$ über 6 Punkten hängt nur von dem auf der $M_1^3$ durch sie gebildeten Wurfe ab.**

Es können nun 6 convergente Ebenen von $K_4$ als $P_i$ genommen werden und um die einer Collineation von $K_4$ entsprechende Transformation in $R_3$ zu bestimmen, ist die Verwandlung der $P_i$ durch die Collineation zu verfolgen. Hier gelten nun die den Theoremen aus Acta Math. XIX, p. 156 analogen Theoreme:

*Theorem CIX.—Wenn die Collineation eine Ebene $P$ ungeändert lässt, ist eine der birationalen Transformationen in $R_3$ äquivalent einer mit 5 Punkten.*

*Jede Collineation gibt in $R_3$ 2 Transformationen, von welchen die eine die Zusammensetzung der anderen mit dem involutorischen Typus 7. Ordnung $Q_7$ ist.*

*Theorem CX.—Alle Transformationen, welche aus einer Collineation an $K_4$ durch Variation des Sextupels $P_1, \ldots . P_6$ erhalten werden, sind äquivalent. Unter ihnen kann man stets auch den Typus erhalten.*

Nun sind die in der cit. Note erhaltenen Collineationen diese:

1. $(\mathfrak{a}\,\mathfrak{a}_1)(\mathfrak{b}\,\mathfrak{b}_1)(\mathfrak{c}\,\mathfrak{c}_1)(\mathfrak{d}\,\mathfrak{d}_1)(\mathfrak{a}_2\,\mathfrak{a}_3)(\mathfrak{b}_2\,\mathfrak{b}_3)(\mathfrak{c}_2\,\mathfrak{c}_3)(\mathfrak{d}_2\,\mathfrak{d}_3)$. 2. $(\mathfrak{a}_1\,\mathfrak{a}_2)(\mathfrak{a}_3\,\mathfrak{b})(\mathfrak{c}\,\mathfrak{d})(\mathfrak{a}\,\mathfrak{b}_3)(\mathfrak{b}_1\,\mathfrak{b}_2)(\mathfrak{c}_3\,\mathfrak{d}_3)(\mathfrak{c}_1)(\mathfrak{c}_2)(\mathfrak{d}_1)(\mathfrak{d}_2)$. 3. $(\mathfrak{a}\,\mathfrak{b}_2\,\mathfrak{b}\,\mathfrak{a}_2)(\mathfrak{a}_1\,\mathfrak{a}_3\,\mathfrak{b}_1\,\mathfrak{b}_3)(\mathfrak{c}\,\mathfrak{d}_1\,\mathfrak{d}\,\mathfrak{c}_1)(\mathfrak{c}_2\,\mathfrak{c}_3\,\mathfrak{d}_2\,\mathfrak{d}_3)$. 4. $(\mathfrak{a}_1)(\mathfrak{a}_2)(\mathfrak{a}_3\,\mathfrak{c})(\mathfrak{b}\,\mathfrak{d})(\mathfrak{a}\,\mathfrak{c}_3)(\mathfrak{b}_1\,\mathfrak{b}_2)(\mathfrak{b}_3\,\mathfrak{d}_3)(\mathfrak{c}_1)(\mathfrak{c}_2)(\mathfrak{d}_1\,\mathfrak{d}_2)$. 5. $(\mathfrak{a}\,\mathfrak{a}_1\,\mathfrak{c}_2\,\mathfrak{c}_3)(\mathfrak{a}_2\,\mathfrak{a}_3\,\mathfrak{c}\,\mathfrak{c}_1)(\mathfrak{b}\,\mathfrak{b}_2\,\mathfrak{d}_1\,\mathfrak{d}_3)(\mathfrak{b}_1\,\mathfrak{b}_3\,\mathfrak{d}\,\mathfrak{d}_2)$. 6. $(\mathfrak{a}_1)(\mathfrak{a}_2\,\mathfrak{a}_3\,\mathfrak{b}\,\mathfrak{c}\,\mathfrak{d}\,\mathfrak{a}_1)(\mathfrak{b}_1\,\mathfrak{d}_3\,\mathfrak{a}\,\mathfrak{b}_2\,\mathfrak{d}_1)(\mathfrak{b}_3\,\mathfrak{c}_2\,\mathfrak{c}_1\,\mathfrak{c}_3\,\mathfrak{d}_2)$. 7. $(\mathfrak{b}_2)(\mathfrak{a}\,\mathfrak{d}_3\,\mathfrak{c}_1)(\mathfrak{a}_2\,\mathfrak{c}\,\mathfrak{d})(\mathfrak{a}_1\,\mathfrak{a}_3\,\mathfrak{b})(\mathfrak{b}_1\,\mathfrak{d}_1\,\mathfrak{d}_2)(\mathfrak{b}_3\,\mathfrak{c}_2\,\mathfrak{c}_3)$. 8. $(\mathfrak{a}_1)(\mathfrak{a}_2)(\mathfrak{c}_1\,\mathfrak{c}_2)(\mathfrak{a}\,\mathfrak{b}_3\,\mathfrak{c}_3\,\mathfrak{d}_3)(\mathfrak{a}_3\,\mathfrak{b}\,\mathfrak{c}\,\mathfrak{d})(\mathfrak{b}_2\,\mathfrak{d}_1\,\mathfrak{b}_1\,\mathfrak{d}_2)$. 9. $(\mathfrak{a}\,\mathfrak{b}_2\,\mathfrak{d}\,\mathfrak{a}_2\,\mathfrak{a}_3\,\mathfrak{b}_1\,\mathfrak{d}_3\,\mathfrak{a}_1)(\mathfrak{b}\,\mathfrak{c}_1\,\mathfrak{c}_3\,\mathfrak{d}_2\,\mathfrak{b}_3\,\mathfrak{c}_2\,\mathfrak{c}\,\mathfrak{d}_1)$. 10. $(\mathfrak{c}_2)(\mathfrak{a}\,\mathfrak{b}_1\,\mathfrak{d}_3)(\mathfrak{a}_1\,\mathfrak{a}_2\,\mathfrak{a}_3\,\mathfrak{b}\,\mathfrak{c}\,\mathfrak{d})(\mathfrak{b}_3\,\mathfrak{c}_1\,\mathfrak{c}_3\,\mathfrak{d}_1\,\mathfrak{b}_2\,\mathfrak{d}_2)$.

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*Ich bezeichne als projectiven Wurf von $m$ Punkten im binären Gebiete den invarianten Character der $m$ Punkte gegen binäre lineare Transformation.

*Theorem CXI.—Den Collineationen* 2. 4. 6. 7. 8. 10. *entsprechen Collineationen und Transformationen, welche durch Zusammensetzung dieser mit* $Q_7$ *entstehen.*

Denn es bleibt bei diesen ein Punkt fest, dessen Ebenen als $P$ genommen werden können, sodass $P_1, \ldots . P_6$ unter einander transformirt werden, weshalb eine Transformation in $R_3$ keine Fundamentalpunkte besitzen kann.

*Theorem CXII.—Die in CXI genannten Transformationen sind resp. äquivalent* $(a_1 b_2)(a_2 b_1)(a_3 b_4)(a_4 b_3)$ *mit* $d_1 d_2$, $(a_1 b_1)(a_2 b_2)(a_3 b_4)(a_4 b_3)$ *mit* $d_1 d_2$, *dem Typus mit dem Index* 10, *dem Typus des Index* 6, $(a_1 b_2)$, $(a_2 b_3)$, $(a_3 b_4)$, $(a_4 b_1)$ *mit involutorischem Paare, einer Collineation.*

Diese Resultate werden durch wirkliche Uebertragung oder auch durch Aufstellung der sämmtlichen $(a_i ; b_i)^3$ gewonnen, welche sich gemäss LXXV über den 6 Punkten unter Hinsicht auf CVIII aufstellen lassen.

*Theorem CXIII.—Die Collineation* 1. *liefert* $(a_i b_i)^3$ *mit involutorischem Paare und einen Typus der Ordnung* 5.

Denn $(a_i b_i)^3\, i_1 i_2$ ist die einzige Transformation der Ordnung 3, welche gar keine Bedingung der 6 Punkte erfordert und dem entspricht nur die Collineation 1. Die Zusammensetzung mit $Q_7$ gibt eine Transformation der Art des §4, welche sich später als typisch erwiesen wird.

*Theorem CXIV.—Die Collineation* 3. *liefert* $(a_1 b_2)(a_2 b_1)$, $b_3$ *in* $a_4$, $b_4$ *in* $a_3$ *und eine* $Q_5$, *welche selbst typisch ist.*

Denn nur diese Transformation des Index 4 bedarf nach LXXV und LXXXXI die gegenwärtige Figur und die Zusammensetzung mit $Q_7$ kann direct darauf unrückgeführt werden.

*Theorem CXV.—Die Collineation* 5. *liefert* $(a_1 b_1)(a_2 b_2)$, $b_3$ *in* $a_4$, $b_4$ *in* $a_3$ *und eine* $Q_5$, *welche selbst typisch ist.*

Aus demselben Grunde wie CXIV.

*Theorem CXVI.—Die Collineation* 9. *liefert einen Typus der* 5. *Ordnung vom Index* 8.

Von diesem Typus wird in einer späteren Arbeit die Rede sein.—Es wäre auch möglich, die Resultate der §§13–16 aus den Eigenschaften der vor Th. CXI hier aufgeschriebenen Collineationen abzuleiten. So interessant es ist, muss ich es dem Leser überlassen.

## §18.—*Eine neue* 1, 2-*deutige Transformation im* $R_3$.

Alle $M_1^4\, p = 1$ durch 7 Punkte $p_1, \ldots . p_7$ gehen durch einen 8. Punkt $a_8$.

An diese $\infty^3$ Curven knüpft sich eine grosse Reihe von Transformationen, indem man nach meinem Verfahren für die Ebene* auf jeder $M_1^4$ oder unter je zwei $M_1^4$ gleichen Moduls eine eindeutig zu bestimmende eindeutige Correspondenz einrichtet. Die einfachste derselben ist eine involutorische Transformation, welche die in §7. angegebene Characteristik verwirklicht.

*Theorem CXVII.—Wird auf jeder $M_1^4$ die Correspondenz $u' + u \equiv \gamma$ mit $a_8$ als Doppelpunkt bestimmt, so entsteht $(8, \ldots. 8)^{15}$ mit $p_1, \ldots. p_7$ als 8 fachen Fundamentalpunkten und $M_2^4(p_i^3 p_{i+1}^2 \ldots. p_{i+6}^2)$ als Fundamentalflächen.*

Da die $M_2^4(p_1^2 \ldots. p_7^2)$ jede $M_1^4$ in Paaren dieser $u' + u \equiv \gamma$ schneiden, so folgt:

*Theorem CXVIII.—Die $(8.8)^{15}$ transformirt die $\infty^6$ $M_2^4(p_1^2 \ldots. p_7^2)$ unter einander u. zw. jede in sich.*

Wenn $n$ die Ordnung des Ortes der Doppelpunktstripel der $u' + u \equiv \gamma$ und $x$ die Vielfachheit in $p_i$ ist, so folgt, weil der Ort invariant durch $(8, \ldots. 8)^{15}$ ist $15.n - 7.4.x = n$, also $n = 2x$ und die Schnittpunktezahl mit einer $M_1^4$ liefert $4n - 7x = 3$, also $x = 3$, $n = 6$. Auf $M_1^4$ mit Doppelpunkt $d$ fallen 2 Doppelpunkte von $u' + u \equiv \gamma$ nach $d$. Daher:

*Theorem CXIX.—Der Ort der Doppelpunkte von $(8, \ldots., 8)^{15}$ ist eine $M_2^6(p_1^3 \ldots. p_8^3)$, welche die 28 Geraden $p_i p_k$ $(i, k = 1 \ldots. 7)$ einfach und die Kegelspitzencurve des Netzes $M_2^2(p_1 \ldots. p_8)$ als einfache Curve enthält, $D_6$.*

Sind nun $q_1 q_1'$, $q_2 q_2'$, $q_3 q_3'$ drei willkürliche Punktepaare der Involution $(8, \ldots., 8)^{15}$, so mögen die $M_2^4(p_1^2 \ldots. p_7^2)$ durch sie den Ebenen eines Raumes $R_3^1$ linear derart entsprechend gemacht werden, dass 5 derselben 5 Ebenen zugewiesen werden. Dann entspricht jedem Punktepaare $pp'$ von $R_3$ ein Punkt von $R_3'$ und umgekehrt. Ich behaupte nun:

*Theorem CXX.—In der Transformation, welche 4. O. in $R_3$ und 16. O. in $R_3'$ ist und den Ebenen von $R_3'$ die $M_2^4(p_1^2 \ldots. p_7^2 q_1 q_1' q_2 q_2' q_3 q_3')$ entsprechen lässt, entspricht der $D_6$ eine Fläche $Z_{12}$ der 12. O., welche einen 9 fachen Punkt $A$, an den drei andere in drei verschiedenen Richtungen unendlich nahe gerückt sind, und in diesen Richtungen drei 6 fache Gerade durch $A$ besitzt.*

Die $D_6$ wird von einer $M_1^{16}$ des Systemes in 12 freien Punkten getroffen. Die $M_1^4$ resp. durch $q_1 q_1'$, $q_2 q_2'$, $q_3 q_3'$ entsprechen Punkten $A'$, $A''$, $A'''$ da die System-$M_2^4$ sie nicht in freien Punkten treffen. Die übrigen $M_1^4$ treffen $D_6$ in drei Punkten und da sie Geraden durch den dem $p_8$ entsprechenden Punkt $A$

* Acta Math., Bd. XIX. §12. B.

liefern, ist $A$ 9-fach. Die Ebenen der 3 6-fachen Geraden $AA'$, $AA''$, $AA'''$ entsprechen den Punktepaaren $q_1 q_1'$, $q_2 q_2'$, $q_3 q_3'$.

Den ebenen Schnitten von $D_6$ entsprechen Curven 24. O., längs welchen $Z_{12}$ von Flächen 16. O. berührt wird, welche den Ebenen selbst entsprechen, den $p_1 \ldots . p_7$ Flächen 4. O.

§19.—*Ableitung der 7-punktigen Typen aus einer Fläche 12. Ordnung.*

Wird über $p_1 \ldots . p_7$ eine Characteristik einer $(a_i;\ b_i)^3$ oder eines aus solchen zusammengesetzten Fundamentalsystemes construirt, so wird dieselbe die $M_2^4 (p_1^2 \ldots . p_7^2)$, aber auch die $M_2^2 (p_1 \ldots . p_7)$ je unter einander transformiren. Nun bilden sich die $M_2^2$ als Quadrikegel durch $AA'$, $AA''$, $AA'''$ ab, da sie von den $M_1^{16}$ in $(16.2 - 7.4)$: 2 Punktepaaren getroffen werden und $\infty^1$ $M_1^4$ enthalten. Eine $M_2^4$ des $\infty^3$ Systemes wird durch die Characteristik in eine $M_2^4$, welche nicht $q_i q_i'$ enthält, verwandelt, und dieser entspricht im $R_3'$ eine $M_2'$ der Ordnung $(4.16 - 7.2.4.) : 2 = 4$, welche da die $M_2^4$ die drei Fundamental $M_1^4$ in je einem Punktepaare der Involution trifft, die Geraden $A_1 A_1'$, $A_1 A_1''$, $A_1 A_1'''$ doppelt und folglich $A$ dreifach enthält. Die Punkte $q_i q_i'$ waren durch die Characteristik in Punkte $[q_i][q_i']$ übertragen, welchen durch die 2, 1 deutige Transformation 3 Punkte $Q_1' \, Q_2' \, Q_3'$ entsprechen, und diese sind den $\infty^3$ $M_2'$ gemeinsam.

Nun haben die supponirten $(a_i;\ b_i)^3$ oder ihre Zusammensetzungen die Eigenschaft, $D_6$ in sich zu transformiren, also wird die Transformation in $R_3'$ die $Z_{12}$ in sich transformiren.

*Theorem CXXI.—Durch die Transposition des* §18 *werden die Reciprokaltransformationen über* $p_1 \ldots . p_7$ *in birationale Transformationen des* $R_3'$ *übertragen, welche i. A. 4. Ordnung sind und in beiden Systemen* $A$ *zum dreifachen Fundamentalpunkte,* $AA'$, $AA''$, $AA'''$ *zu doppelten Fundamentalgeraden und ausserdem in jedem Systeme drei einfache Fundamentalpunkte* $Q_1'$, $Q_2'$, $Q_3'$ *resp.* $Q_1$, $Q_2$, $Q_3$ *haben.*

Man sieht auch leicht, dass die Schnittcurven 4. O. die Doppelgeraden zu Bisecanten haben, einfach durch die $Q$ gehen und also die $M_2^2$ in je einem Punkte schneiden.

*Theorem CXXII.—Wenn* $q_i q_i'$ *so genommen werden, dass von den Paaren* $[q_i][q_i']$ *eines mit einem der* $q_i q_i'$ *zusammenfällt, so erscheint in* $R_3'$ *eine birationale Transformation* 3. *O., welche eine der Doppelgeraden als doppelte, die beiden anderen* *als einfache Fundamentalgeraden besitzt und zwei Punkte R als einfache Fundamentalpunkte.*

Denn ist $[q_i][q'_i]$ mit $q_1 q'_1$ identisch, so sondert sich ein obiger Umformung von $M'_3$ die Ebene $A A'' A'''$ ab und es bleiben nur $Q'_2 Q'_3$. Die Characteristik wird die Verkettung der $Q'$ mit den $Q$ mit Hilfe der Cyclen der Punkte $q_i q'_i$ bestimmen.

*Theorem CXXIII.— Wenn zwei der Paare $[q_i][q'_i]$ mit zweien der $q_i q'_i$ zusammenfallen, so erscheint in $R'_3$ eine birationale Transformation 2. O., welche zwei der Doppelgeraden als einfache Fundamentalgerade und einen einfachen Fundamentalpunkt Q besitzt.*

Es wird von der Art der Coincidenz von $[q_i][q'_i]$ und $q_i q'_i$ abhängen, wie sich die Fundamentalgeradenpaare der beiden Systeme auf die drei Geraden $A A'$, $A A''$, $A A'''$ vertheilen.

*Theorem CXXIV.— Wenn die 3 Paare $q_i q'_i$ durch die Transformation über $p_1 \dots p_7$ nur unter einander vertauscht werden, liefert die Transposition des* §18 *eine Collineation, welche $Z_{12}$ in sich transformirt.*

In diesem Falle werden eben die $M_2^4$ des $\infty^3$-Systemes unter einander vertauscht. Es ist nun aber gewiss in jedem Falle, wo eine einzige Transformation und nicht eine Gruppe über $p_1 \dots p_6$ vorliegt, möglich, drei Punktepaare $q_i q'_i$ auf diese Art zu wählen, müsste man sie auch alle 3 unendlich nahe annehmen. Daher:

*Theorem CXXV.—Die Auffindung der Typen mit* 7 (*und sogar mit* 6, 5) *Punkten ist identisch mit dem Probleme der Collineationen einer Fläche $Z_{12}$ der in CXX bestimmten Art* (*oder ihrer Degenerationen*) *in sich.*

In der That ist durch die Correspondenz in $D_6$ auch die Transformation über $p_1 \dots p_7$ vollkommen bestimmt, insofern man $(8, \dots 8)^7$ als bekannt betrachtet. Denn da die $M_2^2$ die adjungirten Flächen von $D_6$ sind, so muss nach Zeuthen (Math. Ann. IV) und Nöther (Math. Ann. VIII) die Correspondenz die Schnitte von $D_6$ mit diesen $M_2^2$ unter einander verwandeln. Jeder solche Schnitt bestimmt die $M_2^2$ vollkommen, also bestimmt die Correspondenz die Verwandlung der $M_2^2$ und auch der $M_1^4$; dann aber auf jedem Paare von $M_1^4$ vermöge der Schnitte mit $D_6$ die eindeutigen Correspondenzen u. zw. 2, 4, 6, je nachdem alle $M_1^4$ willkürlich, harmonisch oder äquianharmonisch sind.

*Anmerkung.* Für die Typen n° 17, 18, 19 gibt es noch eine andere, ganz directe Construction. Ebenso wie ich in der Preisschrift II §25* Netze von $Q_2$ eingeführt habe, können lineare $\infty^3$-Systeme von $(a_i; b_i)^3$ verwendet werden, von denen die Fundamentalsysteme fest und der einem festen Punkte $p$ entsprechende Punkt $p'$ variabel im $R_3$ ist. Dann besteht zwischen $p'$ und dem n. Transformirten $p^{(n)}$ eine $(N, 1)$ deutige Verwandtschaft und wenn $(a_1 b_2)(a_2 b_3)(a_3 b_4)$ supponirt ist und $b_1$ als $p$ genommen wird, kann man die Punkte $p'$ verlangen, welche $p^{(n)}$ nach $a_4$ bringen, ein Problem, das sich im $R_3$ auf quadratische Gleichungen reduciren lässt, falls $n \leq 3$.

§20.—*Der Typus n.* 16 *und die übrigen Transformationen mit* 4 *Coincidenzen.*

I. $(a_i b_i)$, $i = 1, \ldots 4$ ist immer involutorisch; die 8 Doppelpunkte bilden mit $a_i$ eine desmische Configuration. Die Involution im $R_{18}$ der $M_2^4\ a_i^2$ hat einen Doppel-$R_8$ und einen Doppel-$R_9$, im $R_{12}$ der $M_2^4$ durch die Kanten $a_i a_k$ einer Doppel-$R_5$ und $R_6$, im $R_5$ der $M_2^3$ durch $a_i$ zwei Doppel-$R_2$.

II. $(a_1 b_1)(a_2 b_2)(a_3 b_4)(a_4 b_3)$ ist periodisch, wenn das Ebenenbüschel um $a_1 a_2$ es ist und der Index ist das Doppelte von dessen Index. Ist der Index 2, so entstehen zwei $C_2$ von Doppelpunkten durch $a_3 a_4$ berührend an $a_3 a_1 a_2$, $a_4 a_1 a_2$. Die Stralen über $a_1 a_2$, $a_3 a_4$ sind mit dem Index der Transformation verwandelt.

III. $(a_1 b_2)(a_2 b_1)(a_3 b_4)(a_4 b_3)$ ist periodisch, wenn die Ebenenbüschel um $a_1 a_2$, $a_3 a_4$ es sind und der Index ist das doppelte kleinste Multiplum ihrer Indices. Ist der Index 2 (oder gibt es einen freien Doppelpunkt), so gibt es eine $M_2^2$ erfüllt von Doppelpunkten, welche durch die Geraden $a_1 a_3$, $a_1 a_4$, $a_2 a_3$, $a_2 a_4$ geht und ausserdem einen Doppel-$R_4$ im Raume $R_5$ der $M_2^3$ durch $a_i$. Jeder Stral über $a_1 a_2$, $a_3 a_4$ ist dann in sich transformirt.†

IV. $(a_1 b_1)(a_2 b_3)(a_3 b_4)(a_4 b_2)$ ist stets periodisch vom Index 6. Auf einem Strale durch $a_1$ gibt es 2 Doppelpunkte $d_1 d_2$, welche mit $a_1 \ldots a_4$ eine desmische Configuration bestimmen, deren übrige zwei Tripel cyclische Tripel der Transformation sind. Die invarianten $M_2^3$ sind: 4 Kegeln, deren Spitzen $d_1$ oder $d_2$ sind und deren Berührungsebenen an $d_i a_1$ auf $a_2 a_3 a_4$ die Geraden $di_1$, $di_2$ schneiden; $2M_2^3$, welche die Geraden $a_1 i_1$, $a_1 i_2$ enthalten und, die eine durch $d_1$,

* Cf. Cr. J., Bd. CXIV, p. 50.

† Es entsteht so der einfachste Fall einer allgemeinen Classe von Transformationen, welche eine lineare Stralencongruenz mit periodischen Projectivitäten in sich verwandeln.

die andere durch $d_2$, gehen. ($d$, $i_1$, $i_2$ sind Doppelpunkt und inv. Paar in $(a_2 b_3)$, $(a_3 b_4)$, $(a_4 b_2)$ der Ebene).*

V. $(a_1 b_2)(a_2 b_3)(a_3 b_4)(a_4 b_1)$ (*Typus n*° 16) ist periodisch vom Index 4, sobald es einen einzigen freien Doppelpunkt $d$ gibt, und hat $\infty^1$ Doppelpunkte in dem Strale aus $d$ über $a_1 a_3$, $a_2 a_4$ und zwei $M_2^2$ durch die Geraden $a_1 a_2$, $a_2 a_3$, $a_3 a_4$, $a_4 a_1$ sind invariant. Es gibt dann eine invariante desmische Configuration, von deren Tetraedern eines $a_1 a_2 a_3 a_4$ ist, eines in 2 Doppelpunkte und ein involutorisches Paar getheilt wird und eines ein cyclisches Quadrupel bildet.

Der Index hängt i. A. von der Transformation der Stralen über $a_1 a_3$, $a_2 a_4$ unter einander ab und ist das Doppelte des Indexes derselben. Durch sie ist V vollkommen bestimmt.

*Theorem CXXVI.—Jede Transformation mit* 4 *Coincidenzen ist durch* $a_1 \ldots . a_4$ *und eine invariante* $M_2^2$ *vollkommen bestimmt. Hiebei ist* $M_2^2$ *nur bei* I *durchaus willkürlich.*

Mit $M_2^2$ ist die Verwandlung der Schnitt-$C_2$ in $a_k a_l a_m$ in die Berührungsebene in $b_i$ gegeben und da die Erzeugenden durch $a_i$ in jene durch $b_i$ verwandelt werden müssen, so ist für die quadratische Transformation zwischen $b_i$ und $a_k a_l a_m$ mindestens ein Punktepaar gegeben, in II, III, IV, V zwei, aber so, dass die 4 Bedingungen abhängig und erfüllbar sind.

Calais, 1895.

* Auch in II und III gibt es invariante Tetraederpaare, welche mit $a_1 a_2 a_3 a_4$ eine desmische Configuration bilden.

# *Theories of the Action of Magnetism on Light.*

By A. B. Basset, M. A., F. R. S.

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1. About five years ago I worked out a theory of the reflection and refraction of light at the surface of a magnetized *transparent* medium* by means of a suggestion of Professor Rowland, which consisted in introducing Hall's effect into the general equations of the electro-magnetic field. I did not, however, at the time attempt to apply the results to the phenomenon of reflection from a magnetized *metal*, for being aware that metallic reflection could not be explained upon the electro-magnetic theory by taking into consideration the conductivity of the metal, and that no satisfactory theories of metallic reflection existed

*Phil. Trans. 1891, A, p. 371; Physical Optics, Chap. XX.

except those in which the ether is regarded as an elastic medium, I was uncertain how to proceed. I subsequently determined to try what could be done by transforming the expressions for the intensities of the reflected waves in the same manner as Cauchy and Eisenlohr transformed Fresnel's sine and tangent formulæ;* and I found that the agreement between the theoretical results thus obtained and the experiments of Kerr† was perfect as far as qualitative results were concerned, and in one case the theory showed the existence of a phenomenon which Kerr had failed to detect, but which was shortly afterwards discovered by Kundt‡ upon repeating Kerr's experiments. The agreement as regards quantitative results was not, however, in all cases, so close as might be desired.

But irrespective of the question of metallic reflection, the theory was not entirely satisfactory, since it required the tangential component of the electro-motive force to be discontinuous at the surface of separation of a magnetized and an unmagnetized medium. I was fully aware of this objection at the time, and endeavored to explain the difficulty by suggesting that possibly the transition from one medium to the other was not abrupt, but that a thin interfacial layer might exist through which there was a rapid but continuous change in the value of the tangential component of the electro-motive force. I must however confess that this explanation savors of a device for evading rather than accounting for a difficulty.

2. Mr. Larmor|| has recently attempted to resuscitate a modification of Maxwell's theory,§ which was proposed in 1879 by Professor Fitz Gerald.¶ Fitz Gerald's theory contains a serious defect owing to the boundary conditions being too numerous; and I accordingly awaited the publication of Mr. Larmor's papers with much interest, in the hope that he had succeeded in overcoming the difficulties to which I have alluded. But in this I was disappointed, for the papers in question, instead of containing a careful mathematical investigation of a phenomenon whose interest is equal to the difficulty of satisfactorily

* Proc. Camb. Phil. Soc., vol. VIII, p. 68.

† Phil. Mag., May 1877; Ibid. March 1878.

‡ Phil. Mag., Oct. 1884; and see Physical Optics, Ch. XX.

|| Brit. Assoc. Rep. 1893, p. 335; Phil. Trans. 1894, A, Pt. II, p. 779.

§ Electricity and Magnetism, Chap. XXI.

¶ Phil. Trans. 1880, p. 691.

accounting for it, consisted for the most part of vague and obscure generalities, which are calculated to envelop the subject in a cloud of mystery rather than to enlighten the understanding. The superficial reader may possibly be impressed with their apparent profundity, but when examined they turn out to be a dry husk without a kernel.

3. The object of the present communication is two-fold. In the first place I shall subject Mr. Larmor's theory to a searching examination for the purpose of exposing its imperfections, and shall show that instead of being an improvement on its predecessors it is open to a variety of additional objections and defects. In the second place I shall show that by means of a slight modification of the fundamental hypothesis, the theory of Rowland and myself may be placed on a perfectly satisfactory basis, and that the difficulty with regard to the discontinuity of the tangential component of the electro-motive force at an interface may be removed.

*The Theory of Fitz Gerald and Larmor.*

4. In Maxwell's general theory of the electro-magnetic field, the electric displacement and the magnetic force are connected by the equations

$$4\pi\dot{f} = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \quad 4\pi\dot{g} = \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \quad 4\pi\dot{h} = \frac{d\beta}{dx} - \frac{d\alpha}{dy}, \tag{1}$$

whilst the electro-kinetic energy $T$ and the electro-static energy $W$ per unit of volume are given by the equations

$$T = \frac{\mu}{8\pi}(\alpha^2 + \beta^2 + \gamma^2), \tag{2}$$

$$W = \frac{2\pi}{K}(f^2 + g^2 + h^2). \tag{3}$$

Fitz Gerald introduces a new vector $\mathfrak{M}$, whose time variation is the magnetic force, so that if $\xi$, $\eta$, $\zeta$ be its components,

$$\dot{\xi} = \alpha, \quad \dot{\eta} = \beta, \quad \dot{\zeta} = \gamma, \tag{4}$$

whence it follows from (1) that

$$4\pi f = \frac{d\zeta}{dy} - \frac{d\eta}{dz}, \quad 4\pi g = \frac{d\xi}{dz} - \frac{d\zeta}{dx}, \quad 4\pi h = \frac{d\eta}{dx} - \frac{d\xi}{dy}, \tag{5}$$

so that the electric displacement is proportional to the curl of $\mathfrak{M}$.

The electro-kinetic energy is therefore proportional to the square of the time variation of $\mathfrak{R}$, whilst the electro-static energy depends upon its space variations. Under these circumstances the quantity $T - W$ is a function of $\xi, \eta, \zeta$ and their differential coefficients, and Fitz Gerald proceeds to apply the Calculus of Variations and the Principle of Least Action to obtain the equations of motion and the boundary conditions.

To account for magnetic action upon light, Fitz Gerald, following Maxwell, assumes that the kinetic energy of the medium contains an additional term depending upon the displacement of certain supposed vortices, which is of the form

$$T = 4\pi C\left(\frac{d\xi}{d\theta}\dot{f} + \frac{d\eta}{d\theta}\dot{g} + \frac{d\zeta}{d\theta}\dot{h}\right), \tag{6}$$

where
$$\frac{d}{d\theta} = \alpha_0\frac{d}{dx} + \beta_0\frac{d}{dy} + \gamma_0\frac{d}{dz} \tag{7}$$

and $\alpha_0$, $\beta_0$, $\gamma_0$ denote the components of the external magnetic force.

Fitz Gerald found that the introduction of this additional term into the kinetic energy created a difficulty with regard to the boundary conditions, which Larmor proposes to deal with in the following manner. The components $\xi, \eta, \zeta$ of the vector $\mathfrak{R}$ satisfy the solenoidal condition $d\xi/dx + d\eta/dy + d\zeta/dz = 0$; consequently their variations are not independent but are subject to the above condition. Larmor therefore introduces into the equation of Least Action the additional term

$$\lambda\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right),$$

where $\lambda$ is an undetermined function of $x$, $y$, $z$. He thus obtains the following equations of motion:

$$\mu\frac{d^2\xi}{dt^2} = \frac{1}{K}\left\{\frac{d}{dz}\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) - \frac{d}{dy}\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)\right\}$$
$$- 8\pi C\frac{d}{d\theta}\left(\frac{d\dot{\zeta}}{dy} - \frac{d\dot{\eta}}{dz}\right) - 4\pi\frac{d\lambda}{dx} \tag{8}$$

with two similar ones. From these equations it follows that $\nabla^2\lambda = 0$.

5. Larmor has not attempted to examine the physical meaning of these equations or to enquire how far they agree with those furnished by other theories; and this is what I shall now do.

From (4) and (5) it follows that (8) may be expressed in the form

$$\frac{da}{dt}=\frac{4\pi}{K}\left(\frac{dg}{dz}-\frac{dh}{dy}\right)-32\pi^2 C\frac{d\dot{f}}{d\theta}-4\pi\frac{d\lambda}{dx}. \qquad (9)$$

With the exception of the additional term $d\lambda/dx$, the above equation, which determines the time variation of the magnetic induction, is of the same form as equation (10) of my paper in the Phil. Trans. 1891, as can be seen by recollecting that for an isotropic medium $B^2=C^2=(\mu K)^{-1}$.

If we eliminate $\lambda$ we shall obtain three equations of the form

$$\frac{d^2f}{dt^2}=\frac{1}{K\mu}\nabla^2 f+8\pi C\frac{d}{d\theta}\left(\frac{d\dot{g}}{dz}-\frac{d\dot{h}}{dy}\right). \qquad (10)$$

These are the equations which determine the electric displacement, and they are of the same form as equations (12) of my paper referred to above, from which it follows that, *so far as the propagation of light is concerned*, both theories lead to the same results. The value of the magnetic force is, however, not the same unless $\lambda=0$; and Larmor has shown that $\lambda$ cannot be zero, but is the potential of a surface distribution upon interfaces.

6. There is now a further point to be considered. The introduction of an additional term into the kinetic energy must necessarily produce an alteration in some of Maxwell's general equations of the electro-magnetic field. We must therefore enquire, which of the various sets of equations are modified? What are the additional terms introduced? Can any experimental evidence be brought forward in support of the existence of the additional terms whose presence is a necessary consequence of the theory? But upon all these interesting and important questions Mr. Larmor preserves an impenetrable silence, and not a single hint is given with regard to their solution.

To examine these questions, we must first recall to mind Maxwell's general equations of the electro-magnetic field. They consist of the following four sets:

$$P = -\frac{dF}{dt} - \frac{d\psi}{dx}, \tag{11}$$

$$P = 4\pi f / K, \tag{12}$$

$$a = \frac{dH}{dy} - \frac{dG}{dz}, \tag{13}$$

$$4\pi\mu\dot{f} = \frac{dc}{dy} - \frac{db}{dz}, \tag{14}$$

where $\mu$ is the magnetic permeability.

Equation (14) cannot be altered, since it is assumed in the theory. Equation (13) is a mere analytical artifice which amounts to introducing a quantity, called by Maxwell the vector potential, whose curl is the magnetic induction. The vector potential is essentially an indeterminate quantity, for if the magnetic induction is given, and $F$, $G$, $H$ be any particular values which satisfy (13), the complete values are $F + d\phi/dx$, etc., where $\phi$ is some function of $(x, y, z, t)$. The indeterminateness of the vector potential is taken account of in (11) by the introduction of the quantity $\psi$, and the ambiguity involved in its employment may be got rid of by its elimination.

Equation (12) is implicitly assumed in the expression for the electro-static energy of the medium, which is supposed to be of the form given by Maxwell. The additional term introduced into the energy for the purpose of explaining electro-optical effects is kinetic and not static.

7. Under these circumstances equation (11) is the only one capable of modification. We shall therefore assume that

$$P = -\frac{dF}{dt} - p_3\dot{g} + p_2\dot{h} - \frac{d\psi}{dx} + L, \tag{15}$$

etc., where $p_1 = 32\pi^3 C a_0$, etc. This hypothesis is equivalent to the introduction of Hall's effect into the general equations of E. M. F., together with additional terms $L$, $M$, $N$.

From (13) and (15) we obtain

$$\frac{da}{dt} = \frac{4\pi}{K}\left(\frac{dg}{dz} - \frac{dh}{dy}\right) - 32\pi^3 C \frac{d\dot{f}}{d\theta} + \frac{dN}{dy} - \frac{dM}{dz}. \tag{16}$$

Comparing this with (9) it follows that

$$-4\pi\frac{d\lambda}{dx} = \frac{dN}{dy} - \frac{dM}{dz}. \tag{17}$$

Let $\Phi$ be a solution of Laplace's equation, and let

$$-4\pi\lambda = \Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}, \tag{18}$$

$$\left.\begin{aligned} L &= z\frac{d\Phi}{dy} - y\frac{d\Phi}{dz}, \\ M &= x\frac{d\Phi}{dz} - z\frac{d\Phi}{dx}, \\ N &= y\frac{d\Phi}{dx} - x\frac{d\Phi}{dy}, \end{aligned}\right\} \tag{19}$$

then

$$\frac{dN}{dy} \quad \frac{dM}{dz} = \frac{d}{dx}\left(\Phi + x\frac{d\Phi}{dx} + y\frac{d\Phi}{dy} + z\frac{d\Phi}{dz}\right) \tag{20}$$

$$= -4\pi\frac{d\lambda}{dx}. \tag{21}$$

Since the right-hand side of (18) is a solution of Laplace's equation, $\nabla^2\lambda = 0$ as ought to be the case; whence the requirements of the theory are satisfied by introducing Hall's effect together with the additional terms $L$, $M$, $N$ whose values are given by (19). There does not, however, appear to be any evidence, experimental or otherwise, which would justify the introduction of these additional terms into the general equations of E. M. F., and unless this question can be satisfactorily answered, the theory is open to objections from which its predecessors are free.

8. In order to examine the boundary conditions, Larmor takes the axis of $z$ perpendicular to the surface of separation which is supposed to be plane, and he obtains a result which by virtue of (4) and (5) is equivalent to the condition that

$$\left(\frac{4\pi g}{K} + 4\pi C\frac{d\beta}{d\theta} - 16\pi^2 C\gamma_0\frac{df}{dt}\right)\delta\xi - \left(\frac{4\pi f}{K} + 4\pi C\frac{d\alpha}{d\theta} + 16\pi^2 C\gamma_0\frac{dg}{dt}\right)\delta\eta - \left(\lambda + 16\pi^2 C\gamma_0\frac{dh}{dt}\right)\delta\zeta \tag{22}$$

should be continuous.

When the direction of propagation of a plane wave of light lies in the plane ($xz$), the continuity of the normal component of electric displacement necessitates the continuity of $\beta$; we are therefore forced for the sake of consistency to suppose that $\alpha$ is continuous. Accordingly both the tangential components of the magnetic force must be continuous, which requires that $\delta\xi$ and $\delta\eta$ should be continuous, and it therefore follows that the coefficients of these quantities must be continuous.

Now, according to Maxwell's theory of the electro-magnetic field,

$$P = 4\pi f/K, \quad Q = 4\pi g/K,$$

whence recollecting that in an unmagnetized medium $C = 0$, and denoting the values of quantities in the latter medium by accented letters, the continuity of the coefficients of $\delta\xi$, $\delta\eta$ lead to the two boundary conditions

$$\left.\begin{aligned} Q + 4\pi C\left(\frac{d\beta}{d\theta} - 4\pi\gamma_0 \frac{df}{dt}\right) &= Q', \\ P + 4\pi C\left(\frac{d\alpha}{d\theta} + 4\pi\gamma_0 \frac{dg}{dt}\right) &= P', \end{aligned}\right\} \tag{23}$$

in other words, *the tangential components of the electro-motive force are discontinuous.*

With regard to the last boundary condition, it follows from Maxwell's theory that the normal component of the magnetic induction must be continuous; whence $\mu\delta\zeta = \mu'\delta\zeta'$, so that we obtain from (22)

$$\mu'\left(\lambda + 16\pi^2 C\gamma_0 \frac{df}{dt}\right) = \mu\lambda'.$$

In an unmagnetized medium $\lambda' = 0$, whence at the surface of separation

$$\lambda = -16\pi^2 C\gamma_0\, df/dt, \tag{24}$$

so that $\lambda$ is the potential of a surface distribution of electricity in the interface whose surface value is determined by (24).

9. Larmor, in summing up the results of his theory on p. 349 of the British Association Report for 1893, says: "It appears therefore that we have here a consistent scheme of equations of reflection and refraction without the necessity of condoning any dynamical difficulties in the process, the result being in all respects implicitly involved in the expression for the energy function of the medium." And again, on p. 359, he says: "But against this procedure" (i. e. my own theory) "there stands the pure assumption as regards the discontinuity of electric force at an interface. The correct boundary conditions would be derived from the modification of Fitz Gerald's procedure, which has been explained above." And in an exceedingly verbose paper in the Philosophical

Transactions for 1894 he returns to the subject (p. 780) and says: "I have elsewhere tried to show that, on a consensus of various reasons, this term, originally given by Maxwell, must be taken as the correct representation of the magneto-optic effect." But the preceding analysis shows that there is no justification for the favorable view with which this theory is regarded by its advocate, for when it is stripped of the verbiage in which it is enveloped and subjected to a searching examination, the theory is not only found to be open to the same objections as the one proposed by Rowland and myself—viz. discontinuity of the tangential component of the E. M. F. at an interface—but possesses the further defect of requiring the introduction of certain additional terms into the general equations of E. M. F. for which there is no apparent justification whatever.

We may further enquire, what is the physical meaning of equations (23)? But to this question Mr. Larmor vouchsafes no reply. The equations are a mere aggregation of symbols, to which no meaning other than an analytical one can apparently be attached.

### *Modification of my Former Theory.*

10. The theory of magnetic action which was developed in my paper in the Phil. Trans. may be placed on a satisfactory basis by a modification of the fundamental hypothesis.

According to Maxwell's general theory of the electro-magnetic field, the two equations which involve the electro-motive force are

$$P = -\dot{F} - d\psi/dx, \tag{1}$$
$$P = \quad 4\pi f/K, \tag{2}$$

the first of which is the analytical expression for the theorem that the electro-motive force round a closed circuit is equal to the rate of decrease of the number of lines of magnetic induction passing through the circuit, whilst the second asserts that when electro-motive force acts upon a dielectric it produces an electric displacement which is proportional to the E. M. F. Now the hypothesis adopted by Rowland was that when light is propagated through a transparent medium which is under the influence of a magnetic field, the latter produces an effect analogous to Hall's effect, which may be represented mathematically by introducing the additional terms $-p_3\dot{g} + p_2\dot{h}$, etc., into equation (1). We may, however, in the absence of any experimental evidence to the contrary, assume that the effect of the magnetic field may be equally well represented by introducing the term $p_3\dot{g} - p_2\dot{h}$ into equation (2), and if this be done it will be found that the difficulty with regard to the discontinuity of the E. M. F. at an interface is removed, and that all the equations furnish a consistent scheme in which none of the fundamental principles of dynamics are violated.

11. For the sake of generality we shall suppose that the medium is a doubly refracting one, in which case the relations between electro-motive force and electric displacement will be given by the equations

$$\left.\begin{aligned} P &= 4\pi f/K_1 + p_3\dot{g} - p_2\dot{h}, \\ Q &= 4\pi g/K_2 + p_1\dot{h} - p_3\dot{f}, \\ R &= 4\pi h/K_3 + p_2\dot{f} - p_1\dot{g}, \end{aligned}\right\} \tag{3}$$

where $p_1 = C\alpha_0$, etc., $\alpha_0$, $\beta_0$, $\gamma_0$ being the components of the external magnetic force and $C$ a constant.

All the other equations are the same as Maxwell's (with the exception that we do not suppose that $F$, $G$, $H$ satisfies the solenoidal condition), and are therefore

$$P = -\dot{F} - d\psi/dx, \text{ etc.,} \tag{4}$$

$$a = \frac{dH}{dy} - \frac{dG}{dz}, \text{ etc.,} \tag{5}$$

$$4\pi\mu f = \frac{dc}{dy} - \frac{db}{dz}, \text{ etc.,} \tag{6}$$

where $\mu$ is the magnetic permeability.

Let
$$\mu K_1 = A^{-2}, \ \mu K_2 = B^{-2}, \ \mu K_3 = C^{-2}, \tag{7}$$

$$\Omega = A^2 \frac{df}{dx} + B^2 \frac{dg}{dy} + C^2 \frac{dh}{dz}, \tag{8}$$

$$\frac{d}{d\omega} = p_1 \frac{d}{dx} + p_2 \frac{d}{dy} + p_3 \frac{d}{dz}. \tag{9}$$

From (4) and (5) we obtain

$$\frac{da}{dt} = \frac{dQ}{dz} - \frac{dR}{dy}.$$

Substituting the values of $Q$ and $R$ from (3), we obtain

$$\frac{da}{dt} = 4\pi\mu\left(B^2 \frac{dg}{dz} - C^2 \frac{dh}{dz}\right) - \frac{d\dot{f}}{d\omega}. \tag{10}$$

This is the equation which connects the magnetic induction with the electric displacement, and it is identical with equation (10) of my paper in the Phil. Trans.

Differentiating (6) with respect to the time, and substituting the values of $da/dt$, etc., from (10), we obtain three equations of the form

$$\frac{d^2f}{dt^2} = A^2\nabla^2 f - \frac{d\Omega}{dx} + \frac{1}{4\pi\mu}\frac{d}{d\omega}\left(\frac{d\dot{g}}{dz} - \frac{d\dot{h}}{dy}\right), \qquad (11)$$

which are exactly the same as equations (12) in the Phil. Trans.

So far as the propagation of light is concerned, the modified theory leads to exactly the same results as the original one.

### *The Boundary Conditions.*

12. Let the plane $x = 0$ be the surface of separation, and let the plane $z = 0$ contain the direction of propagation. Then, as shown in my paper in the Phil. Trans., the continuity of $f$, the normal component of the electric displacement, involves the continuity of $\gamma$, which is one of the tangential components of the magnetic force.

The continuity of $a$, the normal component of the magnetic induction, involves the continuity of $H$; for since none of the quantities are functions of $z$, $a = dH/dy$ from (5). For the same reason it follows from (4) that $R$, which is one of the tangential components of the electro-motive force, must also be continuous, and from the last of (3) this leads to the condition

$$4\pi h/K_3 + p_2\dot{f} - p_1\dot{g} = 4\pi h'/K', \qquad (12)$$

where the accented letters refer to the unmagnetized medium.

The mathematical form of this equation is exactly the same as equation (17) of my paper in the Phil. Trans.; accordingly the modified hypothesis shows that the continuity of the normal component of the magnetic induction leads to the continuity of *one* of the tangential components of the E. M. F.

The two remaining boundary conditions must necessarily be continuity of the other tangential components of magnetic force and electro-motive force. The latter condition is represented by the equation

$$4\pi g / K_2 + p_1 \dot{h} - p_3 \dot{f} = 4\pi g' / K', \tag{13}$$

which is the same as (19) of my paper in the Phil. Trans.

We therefore see that the above modification of the original theory leads to a thoroughly consistent scheme of equations, which furnish equivalent results both as regards the representation of the phenomena in the interior of the medium and also as regards the boundary conditions. At the same time the modified theory is free from the defect of violating Newton's third Law of Motion, since the tangential components of the electro-motive force and the magnetic force are both continuous at an interface.

13. The only remaining point to be considered is the deduction of the foregoing results by means of the Principle of Least Action. Mr. Larmor has argued quite correctly that if the value of the energy function is known, the equations of motion and the boundary conditions must necessarily be capable of being deduced therefrom by the Principle of Least Action; but the fallacy of his argument—a fallacy which pervades the whole of his paper in the Philosophical Transactions for 1894, as well as his Report to the British Association in 1893—is that he seems to imagine that if some form of the energy function is postulated, which is capable of furnishing the correct equations of motion and also a sufficient number of equations for determining all the unknown quantities, this function must necessarily be the correct one. The fallacy of this line of reasoning is at once apparent from the fact that if some particular form of the energy function is postulated which furnishes sufficient equations to determine all the unknown quantities, but at the same time leads to results which violate any of the fundamental principles of dynamics, the theory is dynamically unsound, and the results represent some impossible form of motion. To write down certain mathematical expressions and to perform certain mathematical operations do not constitute a satisfactory theory, unless it can be shown that all the results lead to a consistent scheme in which none of the fundamental principles of dynamics are violated.

Let $T$ and $U$ denote Maxwell's expressions for the electro-kinetic and electro-static energies, so that

$$T = \frac{\mu}{8\pi}(\alpha^2 + \beta^2 + \gamma^2), \tag{14}$$

$$U = 2\pi\mu(A^2 f^2 + B^2 g^2 + C^2 h^2), \tag{15}$$

also let

$$U' = \tfrac{1}{2}\{(p_3\dot{g} - p_2\dot{h})f + (p_1\dot{h} - p_3\dot{f})g + (p_2\dot{f} - p_1\dot{g})h\}, \tag{16}$$

and consider the variation of the energy function

$$\delta \int dt \int (T - U - U')\, d\tau = 0, \tag{17}$$

where $d\tau$ denotes an element of volume.

We have at once

$$\begin{aligned}\delta \int dt \int U' d\tau &= \tfrac{1}{2} \int dt \int \{(p_3\dot{g} - p_2\dot{h})\,\delta f - (p_3 g - p_2 h)\,\delta\dot{f} + \dots\}\, d\tau \\ &= \int dt \int \{(p_3\dot{g} - p_2\dot{h})\,\delta f + (p_1\dot{h} - p_3\dot{f})\,\delta g + (p_2\dot{f} - p_1\dot{g})\,\delta h\}\, d\tau \\ &\quad + \text{an integral depending on the limits of the time.}\end{aligned}$$

Introducing the quantities $\xi$, $\eta$, $\zeta$, where the time variations of these quantities are the magnetic force, and recollecting that from (6)

$$4\pi f = \frac{d\zeta}{dy} - \frac{d\eta}{dz}\,, \text{ etc.,} \tag{18}$$

we obtain

$$\begin{aligned}\delta \int dt \int U' d\tau &= \frac{1}{4\pi} \int dt \int \left\{(p_3\dot{g} - p_2\dot{h})\left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz}\right) + \dots\right\} d\tau \\ &= \frac{1}{4\pi} \int dt \int [\{n(p_1\dot{h} - p_3\dot{f}) - m(p_2\dot{f} - p_1\dot{g})\}\,\delta\xi + \dots]\, dS \\ &\quad + \frac{1}{4\pi} \int dt \int \left\{\frac{d\dot{f}}{d\omega}\,\delta\xi + \frac{d\dot{g}}{d\omega}\,\delta\eta + \frac{d\dot{h}}{d\omega}\,\delta\zeta\right\} d\tau,\end{aligned} \tag{19}$$

where $dS$ is an element of the boundary and $l$, $m$, $n$ its direction cosines.

Working out the variation of the term $T - U$ in the same way, we shall find that the equations which hold good in the interior of the medium are

$$\frac{da}{dt} = 4\pi\mu\left(B^2 \frac{dg}{dz} - C^2 \frac{dh}{dy}\right) - \frac{d\dot{f}}{d\omega}, \tag{20}$$

whilst the surface integral term is

$$\int dt \int [\{4\pi\mu(nB^2 g - mC^2 h) + n(p_1\dot{h} - p_3\dot{f}) - m(p_2\dot{f} - p_1\dot{g})\}\,\delta\xi + \dots]\, dS,$$

which, by virtue of (3) and (7), may be written

$$\int dt \int [(nQ - mR)\,\delta\xi + (lR - nP)\,\delta\eta + (mP - lQ)\,\delta\zeta]\,dS. \tag{21}$$

Equation (20) is identical with (10), so that the proposed energy function leads to the equations which must hold good in the interior of the medium. With regard to the surface integral, we shall suppose as before that the plane $x = 0$ is the surface of separation; accordingly $m = n = 0$, $l = 1$, and the boundary conditions require that the expression

$$R\delta\eta - Q\delta\zeta$$

should be continuous. The continuity of the tangential component of the magnetic force at an interface requires that $\delta\eta$ and $\delta\zeta$ should be continuous, and this leads to the condition that $R$ and $Q$ must be continuous. In other words, the tangential components of the electro-motive force must be continuous.

It is unnecessary to introduce the term $\lambda\,(d\xi/dx + d\eta/dy + d\zeta/dz)$ into the equation of Least Action, since the foregoing analysis shows that $\lambda = 0$.

14. We must now consider the meaning of the additional term $U'$.

From the particular manner in which this term has been introduced, it follows that the energy, which is supposed to be due to magnetic action, is electro-static and not electro-kinetic, and from (3) it follows that

$$U + U' = \tfrac{1}{2}(Pf + Qg + Rh),$$

so that the electro-static energy produced by electro-motive force may still be regarded as equal to half the product of the force into the electric displacement; but the E. M. F. contains certain additional terms which depend upon the superimposed magnetic force. There seems to be some ground for supposing that the additional term in the energy ought to be statical rather than kinetic, owing to the fact that the motion consists of oscillations of about a state of strain produced by magnetic force, and this would necessarily lead to a statical term in the potential energy which arises from the statical effects of magnetic force.

15. If in the expression for $U'$ we substitute the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ in terms of the magnetic force, we obtain

$$U' = \frac{1}{8\pi}\left\{f\frac{d\alpha}{d\omega} + g\frac{d\beta}{d\omega} + h\frac{d\gamma}{d\omega} - \left(f\frac{d}{dx} + g\frac{d}{dy} + h\frac{d}{dz}\right)(p_1\alpha + p_2\beta + p_3\gamma)\right\}.$$

Integrating the last term by parts, it will be found that the volume integral vanishes and we get

$$\int U' d\tau = \frac{1}{8\pi}\int\left(f\frac{d\alpha}{d\omega} + g\frac{d\beta}{d\omega} + h\frac{d\gamma}{d\omega}\right)d\tau - \int(lf + mg + nh)(p_1\alpha + p_2\beta + p_3\gamma)\,dS.$$

The volume integral part is of the same form as one of the expressions suggested by Mr. Larmor,* and it is worthy of note that by integrating with respect to the time this expression can be converted into the one proposed by Fitz Gerald. So far as the internal equations are concerned, it is immaterial which of the three expressions we employ, but the preceding analysis shows that the expression for $U'$ employed in the present paper is the only one yet proposed which leads to a "consistent scheme without the necessity of condoning any dynamical difficulties in the process."

FLEDBOROUGH HALL, HOLYPORT, BERKS, *Nov.* 30, 1895.

* Brit. Assoc. Rep. 1893, p. 347.

## *On the Roots of Bessel- and P-Functions.*

By Edward B. Van Vleck.

In the following article I shall confine my attention to the consideration of Bessel- and $P$-functions which are symmetrical in their properties with respect to the real axis of the complex variable. My first purpose is to give a proof of the theorem that between two successive positive or negative real roots of the Bessel function $J_n$ there lies one and only one root of $J_{n+1}$. It is well known that at least one root of $J_{n+1}$ is included in the interval, and it has been supposed that there is only one. But so far as I am aware, no proof of the latter point has been given, and I am confirmed in the impression by the following statement of Grey and Matthews in their recent "Treatise on Bessel Functions" (1895): "It seems probable that between every pair of successive real roots of $J_n(x)$ there is exactly one real root of $J_{n+1}(x)$. It does not appear that this has been strictly proved; there must in any case be an odd number of roots in the interval."

In section II a similar theorem is proved for contiguous ("benachbarte verwandte") Riemann $P$-functions. When two branches of such functions have a common group of substitutions, the roots of the one will alternate with those of the other in each of the three intervals between their singular points 0, 1, $\infty$. It is also pointed out that this theorem includes the foregoing theorem concerning Bessel functions as a limiting case. In a modified form it is extended in section III to contiguous $P$-functions with any number of branch-points. For a $P$-function with but three branch-points there are further formed in section II series of contiguous functions which may be used like the functions of Sturm to determine the number of its real roots between two given limits.

### I.

1. *Between two successive positive or negative roots of the equation* $J_n = 0$ *there lies one and only one root of* $J_{n+1} = 0$. By $n$ is here understood a real number.

When it is not an integer, the demonstration rests upon the relation

$$J_{n+1}J_{-n}+J_{-(n+1)}J_n=\frac{2\sin(n+1)\pi}{\pi x}. \tag{1}$$

In this case $J_n$ and $J_{-n}$ are independent solutions of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\,\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0, \tag{2}$$

whose singular points are 0 and $\infty$. From this it follows, by application of a theorem of Sturm to be cited later, that any two consecutive positive or consecutive negative roots of the equation $J_n=0$ are separated by one root of $J_{-n}=0$ and only one

The same may be also proved in a very simple manner from the formula

$$\frac{J_{-n}}{J_n}=-\int\frac{2\sin n\pi.dx}{\pi x\,J_n^2}. \tag{3}$$

For this shows that $\dfrac{d\frac{J_{-n}}{J_n}}{dx}$ is of constant sign for all positive or all negative values of $x$. Therefore $\frac{J_{-n}}{J_n}$ either continually increases or continually decreases as $x$ passes from 0 to $\infty$. It follows that $x$ passes alternately through the zeros and infinities of this quotient; that is, the real roots of $J_n$ and $J_{-n}$ succeed each other alternately.

Let now any real root of $J_n$ be substituted in (1). It then becomes

$$J_{n+1}J_{-n}=\frac{2\sin(n+1)\pi}{\pi x}.$$

This shows that $J_{n+1}$ and $J_{-n}$ both change sign an even number of times or both an odd number of times between two consecutive roots of $J_n$. But it has just been demonstrated that $J_{-n}$ has one and only one root in the interval, and hence changes but once. Therefore between two consecutive real roots of $J_n$ there lies an odd number of roots of $J_{n+1}$. The substitution of a real root of $J_{n+1}$ will prove in like manner that between two consecutive roots of $J_{n+1}$ there lies an odd number of roots of $J_n$. These two results can be reconciled only when the roots of $J_n$ and $J_{n+1}$ alternate between $x=0$ and $x=\infty$, which was to be proved.

2. When $n$ is an integer, the above proof fails. For $J_n$ and $J_{-n}$ are no longer independent integrals of the equation (2), but $J_{-n} = (-1)^n J_n$; likewise $J_{-(n+1)} = (-1)^{n+1} J_{n+1}$. At the same time $\sin(n+1)\pi = 0$, and both members of (1) therefore vanish identically. To provide for this exceptional case we will introduce the Bessel functions of the second kind, $Y_n$ and $Y_{n+1}$, in place of $J_{-n}$ and $J_{-(n+1)}$. The relation

$$Y_n J_{n+1} - Y_{n+1} J_n = \frac{1}{x}$$

has then only to be substituted for (1) and the proof is as before. Our theorem is therefore true for all real values of $n$.

## II.

3. Two functions $P\begin{pmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{pmatrix}$ and $P_1\begin{pmatrix} 0 & \infty & 1 & \\ \alpha_1 & \beta_1 & \gamma_1 & x \\ \alpha_1' & \beta_1' & \gamma_1' & \end{pmatrix}$ are said to be *related* when the differences of corresponding exponents, $\alpha - \alpha_1$, $\alpha' - \alpha_1'$, etc., are integers, positive or negative. Place

$$\lambda = \alpha' - \alpha, \quad \mu = \beta' - \beta, \quad \nu = \gamma' - \gamma,$$

and denote by $\lambda_1$, $\mu_1$, $\nu_1$ the corresponding exponent-differences of $P_1$. Since the sum of the six exponents of either function is unity, it follows that $\Sigma|\lambda_1 - \lambda|$ —that is, $|\lambda_1 - \lambda| + |\mu_1 - \mu| + |\nu_1 - \nu|$—is an even number. If it is equal to 2, the two functions are said to be *contiguous.*

Taking now two related $P$-functions, and employing the usual notation $P^{\alpha}$, $P^{\alpha'}$, $P_1^{\alpha_1}$, $P_1^{\alpha_1'}$ for the four fundamental branches which correspond to the exponents $\alpha$, $\alpha'$, $\alpha_1$, $\alpha_1'$, we have according to Riemann the relation

$$(P^{\alpha} P_1^{\alpha_1'} - P^{\alpha'} P_1^{\alpha_1}) x^{-\bar{\alpha}} (1-x)^{-\bar{\gamma}} = R(x), \tag{4}$$

in which $\bar{\alpha}$ denotes the smaller of the two numbers $\alpha + \alpha_1'$ and $\alpha' + \alpha_1$, $\bar{\gamma}$ the smaller of the two numbers $\gamma + \gamma_1'$ and $\gamma' + \gamma_1$, and $R(x)$ is a polynomial.* The degree of the polynomial is $\frac{\Sigma|\lambda_1 - \lambda|}{2} - 1$. We shall here confine ourselves to the consideration of contiguous functions, and the polynomial is therefore to be replaced by a constant.

* See Riemann's collected works, p. 73–4.

In place of the four particular branches used in the last equation, four others, more general in character, may be introduced. The two branches of $P$ may be selected arbitrarily and written in the form

$$y = c_1 P^{a} + c_2 P^{a'}, \ \bar{y} = c_3 P^{a} + c_4 P^{a'},$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are constants. For the branches of $P_1$ we will take

$$y_1 = c_1 P_1^{a_1} + c_2 P_1^{a'_1}, \ \bar{y}_1 = c_3 P_1^{a_1} + c_4 P_1^{a'_1}.$$

$y_1$ and $\bar{y}_1$ will then have the same group of substitutions as $y$ and $\bar{y}$ respectively. Between the four branches there exists the obvious relation

$$y\bar{y}_1 - \bar{y}y_1 = (c_1 c_4 - c_2 c_3)(P^{a} P_1^{a'_1} - P^{a'} P_1^{a_1}),$$

whence

$$y\bar{y}_1 - \bar{y}y_1 = Cx^{\bar{a}}(1 - x)^{\bar{\gamma}}. \tag{5}$$

4. This equation is obviously analogous to (1). When the exponents and the constants $c$ are real, the functions $y$, $y_1$, $\bar{y}$, $\bar{y}_1$ are symmetrical in their properties with respect to the real axis of $x$. The equation may then be used to investigate the relative position of the roots of $y$ and $y_1$. First we observe that $y$ and $\bar{y}$, as branches of the same $P$-function, are solutions of a common linear differential equation of the second order with the singular points 0, 1, $\infty$. Sturm, in his classic memoir in the first volume of Liouville's Journal, p. 106, has proved that the real roots of any two symmetrical solutions of a linear differential equation of the second order with real coefficients will, under certain specified restrictions, alternate with one another. These restrictions, stated in the language of the Theory of Functions of a complex variable, confine the application of the theorem to an interval of the real axis included between consecutive singular points of the differential equation. To apply it to the case before us, the $x$-axis is divided into three intervals $\infty 0$, $0\,1$, $1\,\infty$, in each of which a pair of consecutive roots of $y$ will be separated by a single root of $\bar{y}$. Let now a root of $y$ be substituted in (5). We have

$$-\bar{y}y_1 = Cx^{a}(1 - x)^{\bar{\gamma}}.$$

In this the right-hand member has a constant sign throughout each interval. Hence between two consecutive roots of $y$ in each interval, $\bar{y}$ and $y_1$ must change sign either both an even number of times or both an odd number of times. But $\bar{y}$ changes sign once and only once. It follows that an odd number of roots of $y_1$ lie between the two roots of $y$. Since the same is obviously true when $y_1$ and $y$ are interchanged, we have the theorem:

*The real roots of two contiguous P-functions which are symmetrical in their properties with respect to the real axis of the complex variable, and which have the same group of substitutions, will alternate with each other in each interval of the x-axis included between their singular points* 0, 1 *and* $\infty$.

5. With the aid of this theorem we can derive a series of Sturm functions for the determination of the number of real roots of $y$ between two limits $a$ and $b$. The only restriction upon $a$ and $b$ is that no singular points shall be included between them. Let first some function $y_n$ be selected which is related to $y$ and has no roots between $a$ and $b$. It will be proved later that such a selection is always possible. Between the two functions, $y$, $y_n$, any continuous succession of contiguous functions, $y_1$, $y_2$, etc., are next to be inserted. Then in the series $y$, $y_1$, $\pm y_2$, $\pm y_3$, .... $\pm y_n$, the signs can be so fixed that it shall have, between $a$ and $b$, all the properties of Sturm functions. These are as follows:

(1). When $x$ is placed equal to a root of any of these functions except the first and last, the two contiguous functions have opposite signs.

(2). When $x$ traverses the interval from $a$ to $b$ and passes through a root of $y$, the sign of $\frac{y}{y_1}$ changes always from $-$ to $+$ or always from $+$ to $-$.

(3). $y_n$ does not change sign between $a$ and $b$.

The last of these properties has been ensured by our selection of $y_n$. The second follows from the fact that the roots of $y$ and $y_1$ alternate in the interval between $a$ and $b$. The first can be effected by the proper determination of the signs placed before the successive functions of the series. For this purpose let us consider the relations

$$\left.\begin{aligned} y_i \bar{y}_{i-1} - \bar{y}_i y_{i-1} &= C_i x^{\bar{\alpha}_i}(1-x)^{\bar{\beta}_i}, \\ y_{i+1}\bar{y}_i - \bar{y}_{i+1} y_i &= C_{i+1} x^{\bar{\alpha}_{i+1}}(1-x)^{\bar{\beta}_{i+1}}. \end{aligned}\right\} \tag{6}$$

The constant $C$ in either of these equations, if not already known, can be determined by various methods; for example, by expanding each member of the equation into a series arranged according to powers of $x$ and comparing the leading coefficients of the two series. The right-hand members may therefore be regarded as known. Let us now place $x$ equal to some root of $y_i$. We have then

$$-\frac{y_{i-1}}{y_{i+1}} = \frac{C_i}{C_{i+1}} x^{\bar{\alpha}_i - \bar{\alpha}_{i+1}}(1-x)^{\bar{\beta}_i - \bar{\beta}_{i+1}}. \tag{7}$$

Here the right-hand member does not change sign between $a$ and $b$. If it is positive, $y_{\iota-1}$ and $y_{\iota+1}$ have opposite signs whenever $x$ passes through a root of $y_\iota$; if negative, they have the same sign. In the first case, to secure the desired property for the series, it is only necessary to use the same sign before $\bar{y}_{\iota+1}$ as before $y_{\iota-1}$; in the second case, the opposite sign.

The signs can also be determined without the direct use of equations (6), if the recurring relation between three consecutive functions

$$y_{\iota+1} = m_\iota y_\iota - n_\iota y_{\iota-1} \tag{8}$$

is known. The existence of this relation was proved by Riemann, and may be deduced from the equations (6). It is obvious that $n_\iota$ is equal to the right-hand member of (7), and may therefore be employed to discriminate in regard to the signs.

It remains finally to prove that there are functions $y_n$ which are related to $y$ and have no roots between $a$ and $b$. This may be easily inferred from Professor Klein's investigation, "Ueber die Nullstellen der hypergeometrischen Reihe" in the $37^{\text{th}}$ volume of the Mathematische Annalen. The expression there given for the number of roots of a $P$-function in an interval between two singular points whose exponent-differences are $\mu$ and $\nu$ is $E\left(\frac{\lambda-\mu-\nu+1}{2}\right)+\varepsilon$. $E(p)$ denotes the largest positive integer contained in $p$, and is zero when $p$ is negative; $\varepsilon$ is either zero or unity. If, then, the sum of two exponent-differences is greater than the third, there is at most a single root in the interval between the two corresponding singular points. From the same article it is also clear that this root, if there be one, can be removed by increasing the two exponent-differences each by unity (i. e. by the "lateral attachment" of a half plane to the triangle which he employs in his investigation). Now the exponent-differences may be so selected for $y_n$ as to meet these conditions for that interval which includes $a$ and $b$. For the differences of corresponding exponent-differences of $y_n$ and $y$ (viz. $\lambda_n-\lambda$, $\mu_n-\mu$, $\nu_n-\nu$) are integers. Since the exponent-sum $\Sigma(\alpha)$ for both functions is unity, the sum of these three differences must be an even integer. Subject to this restriction $\lambda_n$, $\mu_n$, $\nu_n$ are at our disposal. They may accordingly be chosen in the manner just specified, and in an infinite number of ways.

6. The existence of our Sturm functions to determine the number of roots of our $P$-function between $a$ and $b$ has thus been established. When $y_n$ has been selected, it is immaterial what chain of intermediate functions is used to connect it with $y$. As was pointed out in my article in the 16th volume of this journal, the exponent-differences of two contiguous functions may be related to each other in any one of three ways. Either first, two corresponding exponent-differences may be the same in both functions, and the third pair differ by 2; or secondly, two exponent-differences of the one function may exceed the corresponding two of the other by a unit, the third pair being equal; or lastly, one exponent-difference may be a unit greater and a second a unit smaller than the corresponding exponent-differences of the other function, the third pair again being equal. Corresponding to these three possibilities there are three types of *regular* series of contiguous functions; either a single exponent-difference increases (or decreases) regularly by two units from function to function; or secondly, two exponent-differences increase each by a unit; or lastly, one exponent-difference increases while a second decreases each by a unit.

A particular case of the series here discussed is found in an article by Hurwitz in the 38th volume of the Mathematische Annalen, and was used by him to determine the number of roots of the hypergeometric series between 0 and 1. The series there employed was $F(\alpha, \beta, \gamma, x)$,

$$F(\alpha+1, \beta+1, \gamma+1, x), \ldots \pm F(\alpha+n, \beta+n, \gamma+n, x),$$

$n$ being sufficiently large to ensure that the last function has no roots between 0 and 1. This series is of the 2d type above mentioned. The signs placed before the functions Hurwitz determined by the recurring relation (8).

7. In concluding this section it may be mentioned that the results of section I are included as a limiting case under those given here. $J_n$ and $J_{-n}$ are the limits of the two fundamental branches $P_n$ and $P_{-n}$ of $P\begin{pmatrix} 0 & \infty & a & \\ +n & \beta_1 & \gamma_1 & x \\ -n & \beta_2 & \gamma_2 & \end{pmatrix}$ when under proper restrictions $a$, $\beta_1$, $\beta_2$, $\gamma_1$ and $\gamma_2$ become infinite and disappear from the differential equation.* The point at infinity then becomes an essential singularity, and the only remaining parameters of the function are the

*See Craig's Linear Differential Equations, p. 193.

exponents $\pm n$ which belong to the origin. Accordingly $J_n$ and $J_{n+1}$, also $J_{-n}$ and $J_{-(n+1)}$, are to be considered as corresponding branches of two contiguous Bessel functions. Therefore the relation (4) passes over into (1), and the theorem concerning the alternation of the roots of $J_n$ and $J_{n+1}$ is thus seen to be a limiting case of the theorem given in §4.

## III.

8. Riemann's $P$-function with three singular points satisfies the differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1}\right)\frac{dy}{dx} + \left(\frac{-\alpha\alpha'}{x} + \frac{\gamma\gamma'}{x-1} + \beta'\beta\right)\frac{y}{x(x-1)} = 0.$$

The general form of a regular linear differential equation of the second order with any number of singular points is

$$\frac{d^2y}{dx^2} + \sum_{i=1}^{i=n}\left(\frac{1-\alpha_i-\alpha_i'}{x-e_i}\right)\frac{dy}{dx} + \left(\frac{\alpha_1\alpha_1'\prod\limits_{i=2}^{i=n}(e_1-e_i)}{x-e_1} + \dots + \frac{\alpha_n\alpha_n'\prod\limits_{i=1}^{i=n-1}(e_n-e_i)}{x-e_n}\right.$$
$$\left. + \alpha_\infty\alpha_\infty' x^{n-2} + A_1x^{n-3} + \dots A_{n-2}\right)\frac{y}{\Pi(x-e_i)} = 0.$$

This differs from the preceding by having $n-2$ constants $A_1, \dots A_{n-2}$, which are independent of the singular points and their exponents. These constants are called by Klein "*accessory parameters.*" By a generalized $P$-function we will understand a solution of this differential equation, and we will denote it in the customary manner by $P\begin{pmatrix} e_1 & & e_n & \infty & & \\ \alpha_1 & \dots & \alpha_n & \alpha_\infty & A_1 \dots A_{n-2}, & x \\ \alpha_1' & & \alpha_n' & \alpha_\infty' & & \end{pmatrix}$. *Two P-functions shall by definition be related when the groups of substitutions for any two corresponding branches are the same and are due to circuits of the variable $x$ around common branch-points $e_1, \dots e_r$.* Corresponding exponents in the two functions therefore differ by integers. But this, though necessary, is by no means a sufficient condition as before in the case of functions with but three branch-points. The group of the function depends upon its accessory parameters as well as upon its exponents, and the former must therefore be properly determined in both functions. But unless the total number of singular points is greater than $r$, the number of parameters is not sufficiently large to permit of a common group. To increase the number of parameters, I used in the article already mentioned certain accessory singular points* which were designated by $\rho_1, \rho_2$, etc. The exponents of each of these points are fixed, having the values 2 and 0, and consequently a circuit of $x$ around one or more of these points will not give rise to a substitution of the group. On the other hand, the location of these points for each function is arbitrary, thus placing the required additional constants at our disposal. The least number of such points is in general $r-2$. The complete symbols for two related functions are therefore

$$P\begin{pmatrix} e_1 \dots\dots e_r & \infty & \\ \alpha \qquad \nu & \xi & \rho_1, \rho_2, \dots\dots \rho_{r-2}, A_1, \dots\dots A_{r-2}, x \\ \alpha' \qquad \nu' & \xi' & \end{pmatrix}$$

and

$$P_1\begin{pmatrix} e_1 \dots\dots e_r & \infty & \\ \alpha_1 \qquad \nu_1 & \xi_1 & \rho_1', \rho_2', \dots\dots \rho_{r-2}', A_1', \dots\dots A_{r-2}', x \\ \alpha_1' \qquad \nu_1' & \xi_1' & \end{pmatrix},$$

the differences of corresponding exponents being integers.

The possibility of such a determination of the accessory points and parameters in the two functions that they will have a common group is here assumed and may be inferred from the fact that the number of parameters in the function is equal to the number of conditions imposed by this requirement. A rigorous demonstration was given in the article last cited for the case where one and hence both functions are defined by equations which permit of algebraic integrals. A strict proof is still needed for the general case.

9. Riemann's relation between related $P$-functions may be extended so as to apply to the functions under consideration. His proof of the relation presupposed only the existence of a common group of substitutions for the two functions, due to circuits of $x$ around common branch-points. The same reasoning, extended to the case of more than three branch-points, will give

$$P^{\alpha} P_1^{\alpha_1'} - P^{\alpha'} P_1^{\alpha_1} = (x-e_1)^{\bar{\alpha}} \dots\dots (x-e_r)^{\bar{\nu}} R(x),$$

in which $R(x)$ is a polynomial. It will be noticed that the accessory points do not enter into the right-hand member of this equation. Two related functions

* These points were there designated as *ganzzahlige Punkte* (p. 17). In his lectures of 1893–4 Klein has since called them *Nebenpunkte*, which I have here translated into *accessory points*.

will again be said to be contiguous when the sum of the absolute values of the differences of the corresponding-exponent-differences, $\Sigma|\lambda_i' - \lambda_i|$, is equal to 2. For contiguous functions the polynomial, being of degree $\frac{\Sigma|\lambda_i' - \lambda_i|}{2} - 1$, reduces to a constant. If now $y, \bar{y}, y_1, \bar{y}_1$ be defined as in §3, but in terms of these new functions, we have

$$y\bar{y}_1 - \bar{y}y_1 = C(x-e_1)^{\bar{\alpha}} \dots (x-e_n)^{\bar{\nu}}. \qquad (9)$$

10. This last equation can now be employed to investigate the relative position of the roots of two corresponding branches of any two contiguous functions, $y$ and $y_1$, the properties of which are symmetrical with respect to the real axis of the variable $x$. The branch-points $e$, the exponents, and accessory parameters are then real, while the accessory points are either real or symmetrically located with respect to the axis. If neither function has an accessory point in the interval $e_i e_{i+1}$ between two of their common branch-points, the alternation of the roots of $y$ with those of $y_1$ throughout the whole interval may be argued as before with the help of the above-mentioned theorem of Sturm. Should, however, either function have accessory points between $e_i$ and $e_{i+1}$, we can go no further than to infer the alternation of the roots of $y$ and $\bar{y}$ for the interval between $e_i$ and the nearest accessory point of either function.

To bridge the entire interval between $e_i$ and $e_{i+1}$, it is necessary to use in place of Sturm's theorem the well-known relation between two solutions of the equation $y'' + py' + q = 0$, viz.:

$$y\frac{d\bar{y}}{dx} - \bar{y}\frac{dy}{dx} = e^{-\int p \cdot dx}.$$

For the case before us

$$p = \frac{1-\alpha-\alpha'}{x-e_1} + \dots + \frac{1-\nu-\nu'}{x-e_r} - \frac{1}{x-\rho_1} \dots - \frac{1}{x-\rho_{r-2}},$$

whence follows

$$y\frac{d\bar{y}}{dx} - \bar{y}\frac{dy}{dx} = (x-e_1)^{\alpha+\alpha'-1} \dots (x-e_r)^{\nu+\nu'-1}(x-\rho_1) \dots (x-\rho_{r-2}).$$

Here it is evident that for values of $x$ close to the two extremities of the interval $e_i e_{i+1}$, the right-hand member will have unlike or like signs according as there is an odd or an even number of accessory points between $e_i$ and $e_{i+1}$. In the first case we find, by supposing $y$ to vanish in the equation, that two roots of $y$

are separated by an even number of roots of $y$; in the second case, they are separated by an odd number of roots of $\bar{y}$.

We return now to equation (9) and substitute in this also a root of $y$. The equation then shows that $\bar{y}$ and $y_1$ have each an odd number of roots or each an even number of roots between the two roots of $y$. Combining this result with that of the last paragraph we obtain our final theorem:

*Given two branches of contiguous generalized P-functions which have a common group of substitutions due to circuits of the variable around common branch-points* $e_1, \ldots . e_r$; *if the two functions are symmetrical with respect to the real axis of the variable, two consecutive real roots of either function in the interval between* $e_\iota$ *and* $e_{\iota+1}$ *will be separated by an even or an odd number of roots of the contiguous function, according as the two roots of the function include between them an odd or an even number of its accessory points.*

# *Ueber Collineationsgruppen an Kummer'schen Flächen.*

VON S. KANTOR.

Das Problem der Collineationsgruppen im $R_3$ ist bekanntlich nicht gelöst. Jedoch geht, wenn auch nicht aus den Vorbereitungen einer systematischen Lösung, welche C. Jordan veröffentlicht hat, immerhin aus den gelegentlich bemerkten Gruppen hervor, dass das Problem in enger Beziehung zu den Flächen 4. O. steht. Da jeder Beitrag zu dieser Theorie willkommen sein muss, hauptsächlich aber, weil ich in §17 vorstehender Abhandlung darauf Bezug nehmen muss, will ich hier jene Gruppen, welche eine Kummersche Fläche $K_4$ reproduciren können, vollständig angeben. Dies ist bisher nicht geschehen. Herr Klein hat in Math. Ann., II. Bd.: "Ueber Liniencomplexe 2. Grades" nur diejenige Gruppe von 16 Collineationen gegeben, welche bei der allgemeinsten Kummer'schen Fläche auftritt.

## I.

. Als einfachste Bezeichnung der 16 Punkte und äusserst verallgemeinerungsfähig möchte sich folgende empfehlen. Zwei Reihen $i_1 i_2 i_3 i_4$, $k_1 k_2 k_3 k_4$ mod. 4 werden als Indices von Elementen $ik$ benützt. Dann bilden die 16 Sextupel von Punkten, für welche weder $i$ noch $k$ gleich denen eines festen unter den 16 Elementen, noch das $i+k$ congruent mod. 4 dem $i+k$ des festen Elementes ist, dieselbe Vertheilung in Bezug auf die 16 Punkte, wie die 16 Kegelschnitte in Bezug auf die 16 Doppelpunkte auf $K_4$.* Nimmt man statt $k_1 k_2 k_3 k_4$ die 4

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* Für diese Bezeichnung gibt es eine einfache geometrische Interpretation in der Ebene:

Theorem. Bildet man in Bezug auf drei Punkte $X_1 X_2 X_3$ die Configuration der 16 Punkte $x_i = \sqrt[4]{a_i}$ $(i = 1, 2, 3)$, so ist jeder Punkt mit 6 anderen auf keiner der 12 Geraden; diese 6 Punkte sind in einem Kegelschnitte und diese 16 Kegelschnitte sind stets die Projection der 16 Kegelschnitte einer Kummer'schen Fläche von einem freien Punkte aus.

Buchstaben a, b, c, d, so entsteht die Bezeichnung von Schröter (Cr. J., Bd. C, p. 231), an die ich mich nun halten will.

*Theorem I.—Die 16 Sextupel der Kummer'schen Fläche $K_4$ sind unter einander projectiv.*—Denn die Kleinsche Gruppe ist transitiv und führt auch jeden Kegelschnitt in jeden anderen über.*

*Theorem II.— Wenn eine weitere Collineation P von $K_4$ in sich existirt, so gibt es stets auch eine weitere, welche einen Kegelschnitt in sich überführt.*—Denn führt $P$ den $\alpha$ in $\beta$ über, so erhält man durch Zusammensetzung mit jener Involution, welche $\beta$ in $\alpha$ überführt, $\alpha$ in $\alpha$ und es können nicht 6 Punkte in $\alpha$ fest bleiben, weil damit alle 16 Punkte fest bleiben müssten.

*Theorem III.—Wenn $K_4$, $K_4'$ projective Sextupel haben, so sind sie collinear in einander überführbar.*—Man bedient sich dazu eines Weber'schen Tetraeders jeder $K_4$.† Da zwei Kegelschnitte in den Seitenebenen von $K_4$ zwei Kegelschnitten in denen von $K_4'$ projectiv sind, sodass die beiden gemeinsamen Punkte denen in $R_3'$ entsprechen, so gibt es eine Collineation, welche die 12 Punkte und, da in dem durch das Tetraeder und die Schrötersche $M_3^2$ bestimmten Büschel nur eine $K_4$ verkommt, auch $K_4$ in $K_4'$ überführt.

*Corollar.* Die absoluten Invarianten der $K_4$ stimmen also mit denen der Sextik überein. Hiemit vereinigt sich die Darstellung der $K_4$ durch hyperelliptische Integrale.

*Theorem IV.— Wenn eine Collineation an $K_4$ $\alpha$ in $\beta$ überführt, so ist sie durch die projective Beziehung der beiden Sextupel vollkommen bestimmt.*—Denn durch die Identität in $\alpha$ ist keine andere Raumcollineation als die Identität bestimmt.

*Corollar I.*—Wenn die Collineation $\alpha$ in $\alpha$ überführt, so ist sie durch die projective Beziehung der beiden Sextupel bestimmt.

*Corollar II.* Es gibt so viel Collineationen an $K_4$, welche $\alpha$ in $\beta$ überführen, als Projectivitäten unter den 6 Punkten eines Sextupels von $K_4$ wirklich existiren. Hieraus sofort:

*Theorem V.— Wenn die Sextik der $K_4$ eine Gruppe von k Projectivitäten gestattet so gestattet $K_4 r = 16 . k$ Raumcollineationen in sich.*

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* Das Theorem folgt auch aus dem Satze von W. Stahl, Cr. J., Bd. 98, §8.

† Cr. J., Bd. 86. Diese Tetraeder haben Doppelebenen zu Seiten, aber nicht Doppelpunkte zu Ecken, oder dual.

‡ Cr. J., Bd. C, p. 256.

## II.

Die 15 Involutionen, welche stets existiren, sind:

1. $(\mathfrak{aa}_1)(\mathfrak{bb}_1)(\mathfrak{cc}_1)(\mathfrak{dd}_1)(\mathfrak{a}_2\mathfrak{a}_3)(\mathfrak{b}_2\mathfrak{b}_3)(\mathfrak{c}_2\mathfrak{c}_3)(\mathfrak{d}_2\mathfrak{d}_3)$.
2. $(\mathfrak{aa}_2)(\mathfrak{bb}_2)(\mathfrak{cc}_2)(\mathfrak{dd}_2)(\mathfrak{a}_1\mathfrak{a}_3)(\mathfrak{b}_1\mathfrak{b}_3)(\mathfrak{c}_1\mathfrak{c}_3)(\mathfrak{d}_1\mathfrak{d}_3)$.
3. $(\mathfrak{aa}_3)(\mathfrak{bb}_3)(\mathfrak{cc}_3)(\mathfrak{dd}_3)(\mathfrak{a}_1\mathfrak{a}_2)(\mathfrak{b}_1\mathfrak{b}_2)(\mathfrak{c}_1\mathfrak{c}_2)(\mathfrak{d}_1\mathfrak{d}_2)$.
4. $(\mathfrak{ab})(\mathfrak{a}_1\mathfrak{b}_1)(\mathfrak{a}_2\mathfrak{b}_2)(\mathfrak{a}_3\mathfrak{b}_3)(\mathfrak{cd})(\mathfrak{c}_1\mathfrak{d}_1)(\mathfrak{c}_2\mathfrak{d}_2)(\mathfrak{c}_3\mathfrak{d}_3)$.
5. $(\mathfrak{ac})(\mathfrak{a}_1\mathfrak{c}_1)(\mathfrak{a}_2\mathfrak{c}_2)(\mathfrak{a}_3\mathfrak{c}_3)(\mathfrak{bd})(\mathfrak{b}_1\mathfrak{d}_1)(\mathfrak{b}_2\mathfrak{d}_2)(\mathfrak{b}_3\mathfrak{d}_3)$.
6. $(\mathfrak{ad})(\mathfrak{a}_1\mathfrak{d}_1)(\mathfrak{a}_2\mathfrak{d}_2)(\mathfrak{a}_3\mathfrak{d}_3)(\mathfrak{bc})(\mathfrak{b}_1\mathfrak{c}_1)(\mathfrak{b}_2\mathfrak{c}_2)(\mathfrak{b}_3\mathfrak{c}_3)$.
7. $(\mathfrak{ab}_1)(\mathfrak{a}_1\mathfrak{b})(\mathfrak{dc}_1)(\mathfrak{d}_1\mathfrak{c})(\mathfrak{a}_2\mathfrak{b}_3)(\mathfrak{a}_3\mathfrak{b}_2)(\mathfrak{c}_2\mathfrak{d}_3)(\mathfrak{c}_3\mathfrak{d}_2)$.
8. $(\mathfrak{ab}_2)(\mathfrak{ba}_2)(\mathfrak{cd}_2)(\mathfrak{dc}_2)(\mathfrak{a}_1\mathfrak{b}_3)(\mathfrak{a}_3\mathfrak{b}_1)(\mathfrak{c}_1\mathfrak{d}_3)(\mathfrak{c}_3\mathfrak{d}_1)$.
9. $(\mathfrak{ab}_3)(\mathfrak{a}_3\mathfrak{b})(\mathfrak{cd}_3)(\mathfrak{c}_3\mathfrak{d})(\mathfrak{a}_1\mathfrak{b}_2)(\mathfrak{a}_2\mathfrak{b}_1)(\mathfrak{c}_1\mathfrak{d}_2)(\mathfrak{c}_2\mathfrak{d}_1)$.
10. $(\mathfrak{ac}_1)(\mathfrak{a}_1\mathfrak{c})(\mathfrak{bd}_1)(\mathfrak{b}_1\mathfrak{d})(\mathfrak{a}_2\mathfrak{c}_3)(\mathfrak{a}_3\mathfrak{c}_2)(\mathfrak{b}_2\mathfrak{d}_3)(\mathfrak{b}_3\mathfrak{d}_2)$.
11. $(\mathfrak{ad}_1)(\mathfrak{a}_1\mathfrak{d})(\mathfrak{bc}_1)(\mathfrak{b}_1\mathfrak{c})(\mathfrak{a}_2\mathfrak{d}_3)(\mathfrak{a}_3\mathfrak{d}_2)(\mathfrak{b}_2\mathfrak{c}_3)(\mathfrak{b}_3\mathfrak{c}_2)$.
12. $(\mathfrak{ac}_2)(\mathfrak{a}_1\mathfrak{c}_3)(\mathfrak{bd}_2)(\mathfrak{b}_1\mathfrak{d}_3)(\mathfrak{ca}_2)(\mathfrak{c}_1\mathfrak{a}_3)(\mathfrak{b}_2\mathfrak{d})(\mathfrak{b}_3\mathfrak{d}_1)$.
13. $(\mathfrak{ad}_2)(\mathfrak{a}_1\mathfrak{d}_3)(\mathfrak{bc}_2)(\mathfrak{b}_1\mathfrak{c}_3)(\mathfrak{cb}_2)(\mathfrak{c}_1\mathfrak{b}_3)(\mathfrak{da}_2)(\mathfrak{d}_1\mathfrak{a}_3)$.
14. $(\mathfrak{ac}_3)(\mathfrak{a}_1\mathfrak{c}_2)(\mathfrak{bd}_3)(\mathfrak{b}_1\mathfrak{d}_2)(\mathfrak{ca}_3)(\mathfrak{c}_1\mathfrak{a}_2)(\mathfrak{d}_1\mathfrak{b}_2)(\mathfrak{db}_3)$.
15. $(\mathfrak{ad}_3)(\mathfrak{bc}_3)(\mathfrak{cb}_3)(\mathfrak{da}_3)(\mathfrak{a}_1\mathfrak{d}_2)(\mathfrak{b}_1\mathfrak{c}_2)(\mathfrak{c}_1\mathfrak{b}_2)(\mathfrak{d}_1\mathfrak{a}_2)$.

Die Sextik $(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\,\mathfrak{bcd})$ kann sechs verschiedene Fälle bieten, insofern Projectivitäten in Betracht kommen.*

I. $\mathfrak{a}_1\mathfrak{a}_2$, $\mathfrak{a}_3\mathfrak{b}$, $\mathfrak{cd}$ sind 3 Paare einer Involution. $r = 32$. Die 16 neuen Collineationen sind:

1. $(\mathfrak{a}_1\mathfrak{a}_2)(\mathfrak{a}_3\mathfrak{b})(\mathfrak{cd})(\mathfrak{ab}_3)(\mathfrak{b}_1\mathfrak{b}_2)(\mathfrak{c}_3\mathfrak{d}_3)(\mathfrak{c}_1)(\mathfrak{c}_2)(\mathfrak{d}_1)(\mathfrak{d}_2)$.
2. $(\mathfrak{ab}_2\mathfrak{ba}_2)(\mathfrak{a}_1\mathfrak{a}_3\mathfrak{b}_1\mathfrak{b}_3)(\mathfrak{cd}_1\mathfrak{dc}_1)(\mathfrak{c}_2\mathfrak{c}_3\mathfrak{d}_2\mathfrak{d}_3)$.
3. $(\mathfrak{ab}_1\mathfrak{ba}_1)(\mathfrak{a}_2\mathfrak{a}_3\mathfrak{b}_2\mathfrak{b}_3)(\mathfrak{cd}_2\mathfrak{dc}_2)(\mathfrak{c}_3\mathfrak{d}_1\mathfrak{d}_3\mathfrak{c}_1)$.
4. $(\mathfrak{ab})(\mathfrak{a}_3\mathfrak{b}_3)(\mathfrak{cd}_3)(\mathfrak{c}_1\mathfrak{c}_2)(\mathfrak{c}_3\mathfrak{d})(\mathfrak{d}_1\mathfrak{d}_2)(\mathfrak{a}_1)(\mathfrak{a}_2)(\mathfrak{b}_1)(\mathfrak{b}_2)$.
5. $(\mathfrak{aa}_3)(\mathfrak{a}_1\mathfrak{b}_2)(\mathfrak{a}_2\mathfrak{b}_1)(\mathfrak{b}_3\mathfrak{b})(\mathfrak{c}_1\mathfrak{d}_1)(\mathfrak{c}_2\mathfrak{d}_2)(\mathfrak{c})(\mathfrak{d})(\mathfrak{c}_3)(\mathfrak{d}_3)$.
6. $(\mathfrak{ad}_3\mathfrak{a}_3\mathfrak{d})(\mathfrak{a}_1\mathfrak{c}_2\mathfrak{a}_2\mathfrak{c})(\mathfrak{bc}_3\mathfrak{b}_3\mathfrak{c})(\mathfrak{b}_1\mathfrak{d}_2\mathfrak{b}_2\mathfrak{d}_1)$.
7. $(\mathfrak{ac}_3\mathfrak{a}_3\mathfrak{c})(\mathfrak{a}_1\mathfrak{d}_2\mathfrak{a}_2\mathfrak{d}_1)(\mathfrak{b}_1\mathfrak{c}_2\mathfrak{b}_2\mathfrak{c}_1)(\mathfrak{b}_3\mathfrak{dbd}_3)$.
8. $(\mathfrak{aa}_2\mathfrak{bb}_2)(\mathfrak{a}_1\mathfrak{b}_3\mathfrak{b}_1\mathfrak{a}_3)(\mathfrak{cc}_1\mathfrak{dd}_1)(\mathfrak{c}_2\mathfrak{d}_3\mathfrak{c}_3\mathfrak{d}_2)$.
9. $(\mathfrak{aa}_1\mathfrak{bb}_1)(\mathfrak{a}_2\mathfrak{b}_3\mathfrak{a}_3\mathfrak{b}_2)(\mathfrak{cc}_2\mathfrak{dd}_2)(\mathfrak{c}_1\mathfrak{d}_3\mathfrak{c}_3\mathfrak{d}_1)$.
10. $(\mathfrak{a}_1\mathfrak{b}_1)(\mathfrak{a}_2\mathfrak{b}_2)(\mathfrak{cc}_3)(\mathfrak{c}_1\mathfrak{d}_2)(\mathfrak{c}_2\mathfrak{d}_1)(\mathfrak{dd}_3)(\mathfrak{a})(\mathfrak{b})(\mathfrak{a}_3)(\mathfrak{b}_3)$.
11. $(\mathfrak{ad}_2\mathfrak{b}_3\mathfrak{c}_1)(\mathfrak{a}_1\mathfrak{c}_3\mathfrak{b}_2\mathfrak{d})(\mathfrak{a}_2\mathfrak{cb}_1\mathfrak{d}_3)(\mathfrak{a}_3\mathfrak{d}_1\mathfrak{bc}_2)$.

* Cf. mein Buch: "Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene" (Berlin, Mayer & Müller, 1895).

12. $(\mathfrak{a}\mathfrak{c}_2\mathfrak{b}_3\mathfrak{d}_1)(\mathfrak{a}_1\mathfrak{d}_3\mathfrak{b}_2\mathfrak{c})(\mathfrak{a}_2\mathfrak{d}\mathfrak{b}_1\mathfrak{c}_3)(\mathfrak{a}_3\mathfrak{c}_1\mathfrak{b}\mathfrak{d}_2)$.
13. $(\mathfrak{a}\mathfrak{d}_1\mathfrak{b}_3\mathfrak{c}_2)(\mathfrak{a}_1\mathfrak{c}\mathfrak{b}_2\mathfrak{d}_3)(\mathfrak{a}_2\mathfrak{c}_3\mathfrak{b}_1\mathfrak{d})(\mathfrak{a}_3\mathfrak{d}_2\mathfrak{b}\mathfrak{c}_1)$.
14. $(\mathfrak{a}\mathfrak{c}_1\mathfrak{b}_3\mathfrak{d}_2)(\mathfrak{a}_1\mathfrak{d}\mathfrak{b}_2\mathfrak{c}_3)(\mathfrak{a}_2\mathfrak{d}_3\mathfrak{b}_1\mathfrak{c})(\mathfrak{a}_3\mathfrak{c}_2\mathfrak{b}\mathfrak{d}_1)$.
15. $(\mathfrak{a}\mathfrak{d}\mathfrak{a}_3\mathfrak{d}_3)(\mathfrak{a}_1\mathfrak{c}_1\mathfrak{a}_2\mathfrak{c}_2)(\mathfrak{b}_2\mathfrak{d}_2\mathfrak{b}_1\mathfrak{d}_1)(\mathfrak{b}_3\mathfrak{c}_3\mathfrak{b}\mathfrak{c})$.
16. $(\mathfrak{a}\mathfrak{c}\mathfrak{a}_3\mathfrak{c}_3)(\mathfrak{a}_1\mathfrak{d}_1\mathfrak{a}_2\mathfrak{d}_2)(\mathfrak{c}_1\mathfrak{b}_2\mathfrak{c}_2\mathfrak{b}_1)(\mathfrak{b}_3\mathfrak{d}_3\mathfrak{b}\mathfrak{d})$.

Die Collineationen 1. 4. 5. 10. sind Perspectivitäten, deren Centren und Ebenen dasselbe Tetraeder bilden. Es sind 4 Weber'sche Tetraeder je auf eine Ebene zusammengesunken und $K_4$ ist tetraedroidal. Umgekehrt:

*Theorem VI.—Ist $K_4$ einfach tetraedroidal, so gestattet sie* 2. 16 *Collineationen in sich.*

II. $(\mathfrak{a}_1)(\mathfrak{a}_2)(\mathfrak{a}_3\mathfrak{c})(\mathfrak{b}\mathfrak{d})$ bestimmen eine Involution in $\alpha$. $r = 4.16$. Ausser 32 Collineationen der Art II treten auf:

1. $(\mathfrak{a}_1)(\mathfrak{a}_2)(\mathfrak{c}_1)(\mathfrak{c}_2)(\mathfrak{a}_3\mathfrak{c})(\mathfrak{a}\mathfrak{c}_3)(\mathfrak{b}_1\mathfrak{b}_2)(\mathfrak{b}\mathfrak{d})(\mathfrak{b}_3\mathfrak{d}_3)(\mathfrak{d}_1\mathfrak{d}_2)$.
2. $(\mathfrak{b}_1)(\mathfrak{b}_2)(\mathfrak{d}_1)(\mathfrak{d}_2)(\mathfrak{a}\mathfrak{c})(\mathfrak{a}_1\mathfrak{a}_2)(\mathfrak{a}_3\mathfrak{c}_3)(\mathfrak{c}\mathfrak{d}_3)(\mathfrak{c}_1\mathfrak{c}_2)(\mathfrak{d}\mathfrak{b}_3)$.
3. $(\mathfrak{b})(\mathfrak{d})(\mathfrak{b}_3)(\mathfrak{d}_3)(\mathfrak{a}\mathfrak{a}_3)(\mathfrak{a}_1\mathfrak{c}_1)(\mathfrak{a}_2\mathfrak{c}_2)(\mathfrak{c}\mathfrak{c}_3)(\mathfrak{b}_1\mathfrak{d}_2)(\mathfrak{b}_2\mathfrak{d}_1)$.
4. $(\mathfrak{a})(\mathfrak{c}_3)(\mathfrak{a}_3)(\mathfrak{c})(\mathfrak{a}_1\mathfrak{c}_2)(\mathfrak{a}_2\mathfrak{c}_1)(\mathfrak{b}_1\mathfrak{d}_1)(\mathfrak{b}_2\mathfrak{d}_2)(\mathfrak{b}\mathfrak{b}_3)(\mathfrak{d}\mathfrak{d}_3)$ und 12 Collineationen der Art
5. $(\mathfrak{a}\mathfrak{c}_2\mathfrak{c}_3\mathfrak{a}_1)(\mathfrak{a}_2\mathfrak{a}_3\mathfrak{c}_1\mathfrak{c})(\mathfrak{b}\mathfrak{d}_1\mathfrak{d}_3\mathfrak{b}_2)(\mathfrak{b}_3\mathfrak{d}_2\mathfrak{d}\mathfrak{b}_1)$.

Die Collineationen 1.–4. sind windschiefe Involutionen und es sind resp. $\mathfrak{a}_1\mathfrak{a}_2$, $\mathfrak{c}_1\mathfrak{c}_2$; $\mathfrak{b}_1\mathfrak{b}_2$, $\mathfrak{d}_1\mathfrak{d}_2$; $\mathfrak{b}\mathfrak{d}$, $\mathfrak{b}_3\mathfrak{d}_3$; $\mathfrak{a}\mathfrak{c}_3$, $\mathfrak{a}_3\mathfrak{c}$ die Axen. $K_4$ ist auf zweifache Art Tetraedroid, entsprechend den 8 Weber'schen Tetraedern

$$\begin{array}{cccc} \mathfrak{a}_1 & \mathfrak{a}_2 & \mathfrak{b}_1 & \mathfrak{b}_2 \\ \mathfrak{c}_1 & \mathfrak{c}_2 & \mathfrak{d}_1 & \mathfrak{d}_2 \\ \mathfrak{a} & \mathfrak{a}_3 & \mathfrak{b} & \mathfrak{b}_3 \\ \mathfrak{c} & \mathfrak{c}_3 & \mathfrak{d} & \mathfrak{d}_3 \end{array} \qquad \begin{array}{cccc} \mathfrak{a}_1 & \mathfrak{a}_2 & \mathfrak{d}_1 & \mathfrak{d}_2 \\ \mathfrak{b}_1 & \mathfrak{b}_2 & \mathfrak{c}_1 & \mathfrak{c}_2 \\ \mathfrak{a} & \mathfrak{a}_3 & \mathfrak{d} & \mathfrak{d}_3 \\ \mathfrak{b} & \mathfrak{b}_3 & \mathfrak{c} & \mathfrak{c}_3. \end{array}$$

III. $(\mathfrak{a}_1)(\mathfrak{a}_2\mathfrak{a}_3\mathfrak{b}\mathfrak{c}\mathfrak{d})$ bestimmen eine Projectivität Index 5 in $\alpha$. $r = 5.16$. Die Collineationen sind die 16 gewöhnlichen und 4. 16 von der Art

$$(\mathfrak{a}_1)(\mathfrak{a}_2\mathfrak{a}_3\mathfrak{b}\mathfrak{c}\mathfrak{d})(\mathfrak{b}_1\mathfrak{d}_3\mathfrak{a}\mathfrak{b}_2\mathfrak{d}_1)(\mathfrak{b}_3\mathfrak{c}_2\mathfrak{c}_1\mathfrak{c}_3\mathfrak{d}_2).$$

Die Gruppe ist "unzerlegbar" und sehr beachtenswert.

IV. $(\mathfrak{a}_1\mathfrak{a}_3\mathfrak{b})(\mathfrak{a}_2\mathfrak{c}\mathfrak{d})$ bestimmen eine Projectivität Index 3 in $\alpha$. $r = 6.16$. Die Collineationen sind 16 gewöhnliche, 3. 16 aus I. und 2. 16 der Art

$$(\mathfrak{a}_1\mathfrak{a}_3\mathfrak{b})(\mathfrak{a}_2\mathfrak{c}\mathfrak{d})(\mathfrak{d}_1\mathfrak{d}_2\mathfrak{b}_1)(\mathfrak{b}_3\mathfrak{c}_2\mathfrak{c}_3)(\mathfrak{c}_1\mathfrak{a}\mathfrak{d}_3)(\mathfrak{b}_2).$$

*Theorem VII.—$K_4$ mit* 6. 16 *Collineationen ist dreifach tetraedroidal und die drei Knotentetraeder bilden eine desmische Configuration.*—Denn jedes der 3 Tetraeder muss durch die Perspectivitäten eines anderen Tetraeders in das 3. Tetraeder übertragen werden. Die 12 Weber'schen Tetraeder sind die beiden Quadrupel aus II. und

$$\begin{array}{cccc} \mathfrak{a}_3 & \mathfrak{c} & \mathfrak{d}_2 & \mathfrak{b}_1 \\ \mathfrak{a} & \mathfrak{c}_3 & \mathfrak{d}_1 & \mathfrak{b}_2 \\ \mathfrak{a}_2 & \mathfrak{c}_1 & \mathfrak{b} & \mathfrak{d}_3 \\ \mathfrak{a}_1 & \mathfrak{c}_2 & \mathfrak{b}_3 & \mathfrak{d}. \end{array}$$

*Theorem VIII.—Nimmt man in einer desmischen Configuration auf einer der Linien $h$ (Configurationsgeraden) einen Punkt als* $\mathfrak{a}$, *so bestimmen sich die sämmtlichen übrigen* 16 *Punktedadurch, dass man in den Seitenflächen der Tetraeder vollständige Vierecke construirt welche die Seiten dreiecke der Tetraeder zu Diagonaldreiecken haben.*

V. $(\mathfrak{a}_1)(\mathfrak{a}_3)(\mathfrak{a}_2\mathfrak{b}\mathfrak{c}\mathfrak{d})$ bestimmen eine Projectivität Index 4 in $\alpha$. $r = 24.\ 16$. Ausser 16 gewöhnlichen Collineationen, 6. 16 aus I., 3. 16 aus II., 8. 16 aus IV. sind 6. 16 der folgenden Art vorhanden:

1. $(\mathfrak{a}_1)(\mathfrak{a}_2)(\mathfrak{a}_3\mathfrak{b}\mathfrak{c}\mathfrak{d})(\mathfrak{a}\mathfrak{b}_3\mathfrak{c}_3\mathfrak{d}_3)(\mathfrak{b}_2\mathfrak{d}_1\mathfrak{b}_1\mathfrak{d}_2)(\mathfrak{c}_1\mathfrak{c}_2)$.
2. $(\mathfrak{a}_1\mathfrak{a}\mathfrak{b}_2\mathfrak{d}\mathfrak{a}_2\mathfrak{a}_3\mathfrak{b}_1\mathfrak{d}_3)(\mathfrak{b}\mathfrak{c}_1\mathfrak{c}_3\mathfrak{d}_2\mathfrak{b}_3\mathfrak{c}_2\mathfrak{c}\mathfrak{d}_1)$.
3. $(\mathfrak{a}_1\mathfrak{a}_3\mathfrak{b}_2\mathfrak{d}_3\mathfrak{a}_2\mathfrak{a}\mathfrak{b}_1\mathfrak{d})(\mathfrak{b}\mathfrak{c}_2\mathfrak{c}_3\mathfrak{d}_1\mathfrak{b}_3\mathfrak{c}_1\mathfrak{c}\mathfrak{d}_2)$.
4. $(\mathfrak{a}_1\mathfrak{a}_2)(\mathfrak{a}_3\mathfrak{b}_3\mathfrak{c}\mathfrak{d}_3)(\mathfrak{a}\mathfrak{b}_3\mathfrak{c}\mathfrak{d})(\mathfrak{d}_1\mathfrak{b}_2\mathfrak{d}_2\mathfrak{b}_1)(\mathfrak{c}_1)(\mathfrak{c}_2)$.
5. $(\mathfrak{a}_1\mathfrak{b}_1\mathfrak{c}_2\mathfrak{d}_1)(\mathfrak{a}\mathfrak{a}_3)\mathfrak{a}_2\mathfrak{b}_2\mathfrak{c}_1\mathfrak{d}_2)(\mathfrak{c})(\mathfrak{c}_3)(\mathfrak{b}\mathfrak{d}\mathfrak{b}_3\mathfrak{d}_3)$, 6. ....

Die den 6 Involutionen I. entsprechenden 6 Quadrupel Weberscher Tetraeder welche auf Ebenenquadrupel zusammensinken, sind hier

$$\begin{array}{cccc c cccc c cccc c cccc c cccc c cccc}
\mathfrak{a}_1 & \mathfrak{a}_2 & \mathfrak{b}_1 & \mathfrak{b}_2 & & \mathfrak{a}_1 & \mathfrak{a}_2 & \mathfrak{d}_1 & \mathfrak{d}_2 & & \mathfrak{a}_3 & \mathfrak{c} & \mathfrak{d}_2 & \mathfrak{b}_1 & & \mathfrak{a}_3 & \mathfrak{c} & \mathfrak{d}_1 & \mathfrak{b}_2 & & \mathfrak{b} & \mathfrak{d} & \mathfrak{b}_2 & \mathfrak{d}_2 & & \mathfrak{b} & \mathfrak{d} & \mathfrak{b}_1 & \mathfrak{d}_1 \\
\mathfrak{c}_1 & \mathfrak{c}_2 & \mathfrak{d}_1 & \mathfrak{d}_2 & & \mathfrak{b}_1 & \mathfrak{b}_2 & \mathfrak{c}_1 & \mathfrak{c}_2 & & \mathfrak{a} & \mathfrak{c}_3 & \mathfrak{d}_1 & \mathfrak{b}_2 & & \mathfrak{a} & \mathfrak{c}_3 & \mathfrak{d}_2 & \mathfrak{b}_1 & & \mathfrak{b}_1 & \mathfrak{d}_1 & \mathfrak{b}_3 & \mathfrak{d}_3 & & \mathfrak{b}_2 & \mathfrak{d}_2 & \mathfrak{b}_3 & \mathfrak{d}_3 \\
\mathfrak{a} & \mathfrak{a}_3 & \mathfrak{b} & \mathfrak{b}_3 & & \mathfrak{a} & \mathfrak{a}_3 & \mathfrak{d} & \mathfrak{d}_3 & & \mathfrak{a}_2 & \mathfrak{c}_1 & \mathfrak{b} & \mathfrak{d}_3 & & \mathfrak{a}_2 & \mathfrak{c}_1 & \mathfrak{b}_3 & \mathfrak{d} & & \mathfrak{a} & \mathfrak{c} & \mathfrak{a}_2 & \mathfrak{c}_2 & & \mathfrak{a}_2 & \mathfrak{c}_2 & \mathfrak{a}_3 & \mathfrak{c}_3 \\
\mathfrak{c} & \mathfrak{c}_3 & \mathfrak{d} & \mathfrak{d}_3 & & \mathfrak{b} & \mathfrak{b}_3 & \mathfrak{c} & \mathfrak{c}_3 & & \mathfrak{a}_1 & \mathfrak{c}_2 & \mathfrak{b}_3 & \mathfrak{d} & & \mathfrak{a}_1 & \mathfrak{c}_2 & \mathfrak{b} & \mathfrak{d}_3 & & \mathfrak{a}_1 & \mathfrak{c}_1 & \mathfrak{a}_3 & \mathfrak{c}_3 & & \mathfrak{a}_1 & \mathfrak{c}_1 & \mathfrak{a} & \mathfrak{c}.
\end{array}$$

Indem man sie aus denen von IV zusammensetzt, sieht man:

*Theorem IX.—Die $K_4$ mit* 24. 16 *Collineationen ist* 6-*fach Tetraedroid und die* 6 *Knotentetraeder bilden eine harmonische Configuration* $24_4$.

Durch diese $24_4$ ist aber die $K_4$ bestimmt. Nämlich es gilt:

*Theorem X.—Zu jeder $24_4$ gehören drei 6-fach tetraedroidale $K_4$. In den 4 Seitenflächen eines Tetraeders werden die 4 Punkte erhalten als die Ecken von Vierecken, in deren 6 Seiten die Ebene von den Seitenflächen der zu ihm nicht desmischen Tetraeder geschnitten wird.*

VI. $(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{b}\mathfrak{c}\mathfrak{d})$ bestimmen ein cyclisches Sextupel. $r = 9.16$. Ausser den Collineationen in IV gibt es noch 16 aus I. und 2. 16 der Art:

$$(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{b}\mathfrak{c}\mathfrak{d})(\mathfrak{b}_3\mathfrak{c}_1\mathfrak{c}_3\mathfrak{d}_1\mathfrak{b}_2\mathfrak{d}_2)(\mathfrak{a}\mathfrak{b}_1\mathfrak{d}_3)(\mathfrak{c}_2).$$

$K_4$ ist vierfach tetraedroidal.

In jedem dieser 6 Fälle gestattet $K_4$ die gleiche Anzahl von Correlationen in sich wie von Collineationen. Cf. mein allgemeines Theorem Cr. J., Bd. CXVI.

CALAIS (Reims) 1895.

# *Note on Linear Differential Equations with Constant Coefficients.*

By F. Franklin.

1. In order to solve the differential equation

$$a_0 \frac{d^m y}{dx^m} + a_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots + a_m y \equiv f\left(\frac{d}{dx}\right) y = 0, \tag{1}$$

we may put $y = ve^{ax}$, $v$ being an undetermined function of $x$, as does M. Picard in his Traité d'Analyse, vol. III, p. 392. The result of this substitution in the first member of the equation is

$$e^{ax}\left[f(a)\,v + f'(a)\frac{dv}{dx} + \ldots + f^{(p)}(a)\frac{d^p v}{dx^p} + \ldots\right], \tag{2}$$

and hence, if $a$ is a $p$-fold root of the algebraic equation $f(z) = 0$, it is plain that the given differential equation will be satisfied by putting

$$v = C_0 + C_1 x + \ldots + C_{p-1} x^{p-1}, \tag{3}$$

where the $C$'s are arbitrary constants.

2. The solutions thus obtained, and corresponding to the various roots $a_1, a_2, \ldots a_n$ of the equation $f(z) = 0$, appear to constitute a complete system of solutions of the differential equation, since they comprise $m$ particular solutions of the type $x^q e^{ax}$; but in order to prove that the system is really complete, it is necessary to show that there can exist no linear relation of the form

$$P_1 e^{a_1 x} + P_2 e^{a_2 x} + \ldots + P_n e^{a_n x} \equiv 0, \tag{4}$$

where the $P$'s are polynomials in $x$. M. Picard proves this in a manner sufficiently elementary, but somewhat artificial and involving considerable detail. The impossibility of the relation (4) may, however, be shown instantaneously and without any calculation.

3. We have only to observe that if $v$ be a polynomial of the degree $p-1$, so that $v, \frac{dv}{dx}, \ldots.$ are polynomials of descending degrees, the expression (2) does not vanish unless $f(\alpha) = f'(\alpha) = \ldots. = f^{(p-1)}(\alpha) = 0$, i. e. unless $\alpha$ is a $p$-fold root of $f(z) = 0$. Hence, in order that $y = Pe^{\alpha x}$ be a solution of a differential equation (linear and with constant coefficients), $P$ being a polynomial of degree $p-1$, it is *necessary* as well as sufficient that $\alpha$ be a $p$-fold root of the auxiliary algebraic equation of that differential equation. Now $y = P_2 e^{\alpha_2 x} + \ldots. + P_n e^{\alpha_n x}$ is evidently a solution of a certain differential equation whose auxiliary algebraic equation does not possess the root $\alpha_1$; but, by what has just been said, $y = P_1 e^{\alpha_1 x}$ cannot be a solution of this last-named equation, and therefore cannot be identical with $-(P_2 e^{\alpha_2 x} + \ldots. + P_n e^{\alpha_n x})$, so that the relation (4) cannot exist. Q. E. D.

# *On Certain Partial Differential Equations connected with the Theory of Surfaces.**

BY THOMAS CRAIG.

Let $u$ and $v$ denote the parameters of the lines of curvature of a surface, $\rho_1$ and $\rho_2$ the principal radii of curvature of the surface at the point $(u, v)$, $\rho_1$ being the principal radius corresponding to the line $v =$ const. (the $u$-line), and $\rho_2$ corresponding to $u =$ const. (the $v$-line). Denote further by $R_1$, $R_2$ the radii of geodesic curvature of the lines $u =$ const. and $v =$ const. respectively; let $\alpha$, $\beta$, $\gamma$ be the direction cosines of the normal to the surface and write

$$L = \sum \alpha \frac{\partial^2 x}{\partial u^2}, \quad M = \sum \alpha \frac{\partial^2 x}{\partial u \partial v} = 0, \quad N = \sum \alpha \frac{\partial^2 x}{\partial v^2}. \tag{1}$$

We have now

$$\left.\begin{aligned} \rho_1 &= \frac{E}{L}, \quad \rho_2 = \frac{G}{N}, \\ \frac{1}{R_2} &= \frac{-1}{2E\sqrt{G}} \frac{\partial E}{\partial v}, \quad \frac{1}{R_1} = \frac{-1}{2G\sqrt{E}} \frac{\partial G}{\partial u}. \end{aligned}\right\} \tag{2}$$

We have further the following three equations connecting all of the preceding quantities

$$\left.\begin{aligned} \frac{1}{\rho_1^2} \frac{\partial \rho_1}{\partial v} + \frac{\sqrt{G}}{R_2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = 0, \\ \frac{1}{\rho_2^2} \frac{\partial \rho_2}{\partial u} - \frac{\sqrt{E}}{R_1}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = 0, \end{aligned}\right\} \tag{3}$$

$$\frac{\sqrt{EG}}{\rho_1 \rho_2} = \frac{\partial}{\partial u} \frac{\sqrt{G}}{R_1} + \frac{\partial}{\partial v} \frac{\sqrt{E}}{R_2}. \tag{4}$$

Equations (3) can also be written

$$\left.\begin{aligned} \frac{\partial}{\partial v} \frac{\sqrt{E}}{\rho_1} &= \frac{\sqrt{E}}{R_2} \cdot \frac{\sqrt{G}}{\rho_2}, \\ \frac{\partial}{\partial u} \frac{\sqrt{G}}{\rho_2} &= -\frac{\sqrt{G}}{R_1} \cdot \frac{\sqrt{E}}{\rho_1}. \end{aligned}\right\} \tag{5}$$

---

* "Sur une suite d'équations linéaires aux dérivées partielles provenant de la théorie des surfaces." Comptes Rendus, Oct. 26, 1896, p. 634. In this note the letter $\varepsilon$ has been printed throughout for $E$.

Differentiating the first of (3) for $u$ and the second for $v$ and eliminating $\frac{1}{\rho_2}$ and $\frac{1}{\rho_1}$ respectively, we have

$$\frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_1} - \frac{\sqrt{G}}{R_2}\frac{\partial}{\partial u}\frac{1}{\rho_1} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial}{\partial v}\frac{1}{\rho_1} = 0, \tag{6}$$

$$\frac{\partial^2}{\partial u \partial v}\frac{1}{\rho_2} - \left(\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}\right)\frac{\partial}{\partial u}\frac{1}{\rho_2} - \frac{\sqrt{E}}{R_1}\frac{\partial}{\partial v}\frac{1}{\rho_2} = 0; \tag{7}$$

that is, $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are particular integrals of the differential equations

$$(E_{01}) = \frac{\partial^2\phi_{01}}{\partial u \partial v} - \frac{\sqrt{G}}{R_2}\frac{\partial\phi_{01}}{\partial u} - \left(\frac{\partial}{\partial u}\log\frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1}\right)\frac{\partial\phi_{01}}{\partial v} = 0, \tag{8}$$

$$(E_{02}) = \frac{\partial^2\phi_{02}}{\partial u \partial v} - \left(\frac{\partial}{\partial v}\log\frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2}\right)\frac{\partial\phi_{02}}{\partial u} - \frac{\sqrt{E}}{R_1}\frac{\partial\phi_{02}}{\partial v} = 0. \tag{9}$$

The differential equation satisfied by the coordinates $(x, y, z)$ of the point $(u, v)$ of the surface is

$$\frac{\partial^2\phi_0}{\partial u \partial v} + \frac{\sqrt{G}}{R_2}\frac{\partial\phi_0}{\partial u} + \frac{\sqrt{E}}{R_1}\frac{\partial\phi_0}{\partial v} = 0. \tag{10}$$

From equations (5) we derive in the same way as above the following:

$$\frac{\partial^2\phi_{01}}{\partial u \partial v} - \frac{\partial}{\partial u}\log\frac{\sqrt{E}}{R_2}\cdot\frac{\partial\phi_{01}}{\partial v} - \frac{\sqrt{EG}}{R_1R_2}\phi_{01} = 0, \tag{11}$$

$$\frac{\partial\phi_{02}}{\partial u \partial v} - \frac{\partial}{\partial v}\log\frac{\sqrt{G}}{R_1}\cdot\frac{\partial\phi_{02}}{\partial u} - \frac{\sqrt{EG}}{R_1R_2}\phi_{02} = 0. \tag{12}$$

These are *equivalent* to (8) and (9) (that is, have the same invariants, or are derived from (8) and (9) by substitutions of the form $\phi = \lambda\phi'$) and have $\frac{\sqrt{E}}{\rho_1}$ and $\frac{\sqrt{G}}{\rho_2}$ respectively as particular integrals.

The invariants of $(E_{01})$ are

$$h_{01} = \frac{\sqrt{EG}}{R_1R_2}, \quad k_{01} = -\frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{E}}{R_2} + \frac{\sqrt{EG}}{R_1R_2}; \tag{13}$$

those of $(E_{02})$ are

$$h_{02} = -\frac{\partial^2}{\partial u \partial v}\log\frac{\sqrt{G}}{R_1} + \frac{\sqrt{EG}}{R_1R_2}, \quad k_{02} = \frac{\sqrt{EG}}{R_1R_2}. \tag{14}$$

Form now the Laplace's series,

$$\left.\begin{matrix} \ldots.(E_{-i1}), \ldots.(E_{-21}), (E_{-11}), (E_{01}), (E_{11}), (E_{21}) \ldots., (E_{i1}) \ldots. \\ \ldots.(E_{-i2}), \ldots.(E_{-22}), (E_{-12}), (E_{02}), (E_{12}), (E_{22}) \ldots., (E_{i2}) \ldots. \end{matrix}\right\} \tag{15}$$

It is easy to see by calculation of the invariants that these two series of equations are equivalent, viz. the invariants of $(E_{k-1,2})$ and $(E_{k,1})$ are the same, or, what comes to the same thing, the equations in the lower line are transformed into those in the upper line by substitutions of the form $z = \lambda z'$, denoting by $z$ the unknown function and by $\lambda$ a function of $(u, v)$. In $(E_{k-1,2})$ replace $\phi_{k-1,2}$ by $\frac{\sqrt{G}}{R_2}\phi'_{k-1,2}$, the new equation in $\phi'_{k-1,2}$ will be $(E_{k,1})$, that is, $\phi'_{k-1,2} \equiv \phi_{k1}$. This is for $k$ positive. For the negative values of $k$ we write for $\phi_{-k2}$ the quantity $\frac{\sqrt{E}}{R_1}\phi'_{-k2}$ and so transform $(E_{-k2})$ into $(E_{-(k-1),1})$.

We find readily the following relations for the invariants:

$$\left.\begin{aligned} h_{i1} = h_{i-1,2} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{G^{\frac{i}{2}}}{R_1^i} h_{11}^{i-1} h_{21}^{i-2} \dots h_{i-1;1}\right), \\ k_{i1} = k_{i-1,2} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{G^{\frac{i-1}{2}}}{R_1^{i-1}} h_{11}^{i-2} h_{21}^{i-3} \dots h_{i-2,1}\right), \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} h_{-i2} = h_{-(i-1),1} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{E^{\frac{i-1}{2}}}{R_2^{i-1}} k_{-12}^{i-2} k_{-22}^{i-3} \dots k_{-(i-2),2}\right), \\ k_{-i2} = k_{-(i-1),1} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{E^{\frac{i}{2}}}{R_2^i} k_{-12}^{i-1} k_{-22}^{i-2} \dots k_{-(i-1),2}\right), \end{aligned}\right\} \tag{17}$$

Dropping the letter $k$ and using only $h$ (i. e. $h_{-1} = k$), these are

$$h_{i-1,2} = h_{i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{G^{\frac{i}{2}}}{R_1^i} h_{11}^{i-1} h_{21}^{i-2} \dots h_{i-1,1}\right), \tag{18}$$

or

$$h_{i-1,2} = h_{i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{G^{\frac{i}{2}}}{R_1^i} h_{02}^{i-1} h_{12}^{i-2} \dots h_{i-2,2}\right). \tag{18'}$$

Also

$$h_{-(i+1),2} = h_{-i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{E^{\frac{i}{2}}}{R_2^i} h_{-11}^{i-1} h_{-21}^{i-2} \dots h_{-(i-1),2}\right), \tag{19}$$

or

$$h_{-(i+1),2} = h_{-i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log\left(\frac{E^{\frac{i}{2}}}{R_2^i} h_{-22}^{i-1} h_{-32}^{i-2} \dots h_{-i2}\right). \tag{19'}$$

Before considering the general case, we shall examine the case where equations (8) and (9) become identical. For this we must have

$$\frac{\partial}{\partial u} \log \frac{\sqrt{G}}{R_2} = 0, \quad \frac{\partial}{\partial u} \log \frac{\sqrt{E}}{R_1} = 0. \tag{20}$$

From these we find readily

$$E = U_1 V_2, \quad G = U_2 V_1, \tag{21}$$

the $U$'s being functions of $u$ alone and the $V$'s functions of $v$ alone. We have now for the linear element

$$ds^2 = U_2 V_2 \left( \frac{U_1}{U_2} du^2 + \frac{V_1}{V_2} dv^2 \right), \tag{22}$$

or say

$$ds^2 = \frac{1}{\lambda} (U du^2 + V dv^2). \tag{23}$$

In this case then the lines of curvature form an isometric system. The differential equation satisfied by $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ is now readily seen to take the form

$$\frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial v} \frac{\partial \phi}{\partial u} - \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \phi}{\partial v} = 0, \tag{24}$$

and that satisfied by the coordinates $x$, $y$, $z$ of a point on the surface is

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \theta}{\partial v} = 0. \tag{25}$$

This is the adjoint equation to (24), since $\lambda$ is of the form $\frac{1}{U_2(u) \, V_2(v)}$, or say simply $U_2 V_2$, for now $\frac{\partial^2 \log \lambda}{\partial u \partial v} = 0$.

In the case then *when the lines of curvature of a surface form an isometric system, and the linear element has the form*

$$G^2 = U_2(u) \, V_2(v)(U du^2 + V dv^2),$$

*the partial differential equation satisfied by the reciprocals of the principal radii of curvature is adjoint to that satisfied by the cartesian coordinates of a point of the surface.*

The common value of the invariants of both equations is

$$h = k = \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial v} = \tfrac{1}{4} \frac{U_2'(u)}{U_2(u)} \cdot \frac{V_2'(v)}{V_2(v)}. \tag{26}$$

It is not my purpose here to examine the general system of equations implied in (15), but we may look for a moment at the first of the Laplace's equations derived from (24); this is

$$\frac{\partial^2 \phi_1}{\partial u \partial v} - \frac{\partial \log \lambda^{\frac{1}{2}} h}{\partial v} \frac{\partial \phi_1}{\partial u} - \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \phi_1}{\partial v} + \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \log h}{\partial v} \phi_1 = 0. \tag{27}$$

The invariants are

$$\left.\begin{aligned} h_1 &= \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial v} - \frac{\partial^2 \log h}{\partial u \partial v} \\ &= h - \frac{\partial^2 \log h}{\partial u \partial v}, \\ k_1 &= h. \end{aligned}\right\} \qquad (28)$$

But

$$h = \tfrac{1}{4} \frac{\partial \log \lambda}{\partial u} \frac{\partial \log \lambda}{\partial v} = \tfrac{1}{4} \frac{U_2'}{U_2} \cdot \frac{V_2'}{V_2};$$

therefore

$$\frac{\partial^2 \log h}{\partial u \partial v} = 0,$$

and consequently

$$h_1 = k_1 = h = k.$$

The same thing will clearly be true of all the following equations of the positive end of Laplace's series, and from the known properties of this series in the case when the leading equation has equal invariants we shall have all the invariants of the negative end of the series equal and equal to the single invariant of the leading equation. All the equations of Laplace's series are therefore equivalent in this case, a fact which is indeed almost evident *à priori*. An extended study of equations (8) and (9) and the corresponding Laplace's series (15) will be undertaken in another paper.

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# SYLVESTER.

### 1814—1897

Professor Sylvester, who died in London on Monday, March 15, 1897, was the principal founder of the American Journal of Mathematics. He was its editor from its foundation in 1877 until he left the United States in December, 1883; in this editorial service he had the assistance of successive associate editors. During the first ten years of the existence of the Journal he contributed to its pages upwards of fifty papers, the greater number of these being contained in the first seven volumes. It is undoubtedly owing to his interest and enthusiasm that the Journal has attained its position in the scientific world.

James Joseph Sylvester was born in London, September 3, 1814. He was educated at two private schools in London and at the Royal Institution, Liverpool, whence he proceeded to St. John's College, Cambridge. He was Second Wrangler in 1837, and received the degrees of B. A. and M. A. from Dublin University in 1841 and from the University of Cambridge in 1872; he became a student of the Inner Temple, July 29, 1846, and was called to the Bar, November 22, 1850. He was for some time Professor of Natural Philosophy at University College, London, and Professor of Mathematics at the University of Virginia from 1840 to 1841. After his return to England he was for ten years engaged with a firm of actuaries. In 1855 he was appointed Professor of Mathematics at the Royal Military Academy, Woolwich, which position he held up to 1870.

After an interval of a little more than five years, he was invited, at the opening of the Johns Hopkins University, in 1876, to become its Professor of Mathematics, which position he held until December, 1883, when he was elected Savilian Professor of Geometry at Oxford, succeeding Professor Henry Smith.

He was elected a Fellow of the Royal Society in 1839, received a Royal Medal in 1860, and the Copley Medal in 1880. He was a Corresponding Member of the Institute of France, of the Imperial Academy of Science of St. Petersburg, of the Royal Academy of Science of Berlin, of the Lincei of Rome, of the Istituto Lombardo, and of the Société Philomatique. He was a Fellow of New College, Oxford; Foreign Associate of the United States National Academy of Sciences; Foreign Member of the Royal Academy of Sciences, Göttingen, of the Royal Academy of Sciences, Naples, and of the American Academy of Arts and Sciences, Boston. In 1890 the "Décoration d'Officier de la Légion d'Honneur" was conferred upon him by the President of the French Republic. He received honorary degrees from the Universities of Oxford, Cambridge, Dublin and Edinburgh.

[*Supplement to the American Journal of Mathematics, April 1897.*]

2. The forms of magic squares most familiarly known have been symmetrical, though not pandiagonal. It is not to be presumed that symmetrical squares should have been known so long, possessing the property, for the square of 7, that two centrally opposite numbers shall have 50 for their sum, and in general, for the square of $2n+1$, that two centrally opposite numbers shall have $(2n+1)^2+1$ for their sum, without some one noticing that any $n$ numbers, plus their opposites, plus the central number, making $2n+1$ in all, must form a symmetrical group having the same sum as that of any row or column. Granting that this observation must have been made, unknown to the present writer, it would yet appear that this property of summing symmetrical groups has been slighted beyond measure and without cause. In A, for example, there are 28 ways of summing 175 by rows, columns, and diagonals, while by reference to the symmetric groups hundreds upon hundreds of other ways may be exhibited. If dealing with trifles like magic squares is worthy of an Euler or a Cayley; if in short it is legitimate, it is because amusement has value. Any one who writes a symmetrical square indelibly upon some substance permitting instant removal of temporary marks can excite much interest by marking the numbers composing a symmetrical group, and subsequently exhibiting in like manner many other such groups, all having the same sum. It will be assumed in this paper that symmetry is essential in every case.

3. Pandiagonal squares, not symmetrical, have also been known for centuries. It is a property of pandiagonal squares, long known and easily recognized, that a row or column on one of the four sides can be transposed to the opposite side without destroying the magic and pandiagonal attributes of the square. As the operation may be repeated indefinitely, any number of rows or columns taken together can be transposed with the same freedom. By reason of these remarkable properties pandiagonal squares have received from some writers, beginning with La Hire, the name "perfect," and from others, beginning with Lucas, the name "diabolic." It is therefore possible to bring any given number in such a square, by not more than two transpositions, into the middle or any other given position. The square A, for example, is only one of 49 commutative forms, and of each of these forms there are 8 varieties, because any square whatever may be turned upside down or sideways, or reflected in a mirror, without losing its identity. The other 48 forms are not symmetrical, but any one of them may be regarded as "capable of symmetry," requiring only such transpo-

sitions as shall bring the middle number to the middle place. In most cases pandiagonal squares are not capable of symmetry.

4. A special property of pandiagonal squares has been variously stated by analysts. The following description will probably be found as simple as the matter permits of. Choose two numbers in the square, certain combinations excepted, and note their relative location, observing how many places up or down, how many to the right or left, the second number stands from the first, and call this measurement the "step" between the two. It is forbidden to choose two numbers which, when divided severally by the root of the square, that is, the number of places on its side, say $r$, shall have either the same integral quotient or the same remainder; provided, as regards the division, that when there is no remainder the quotient must be reduced by 1, leaving $r$ as the remainder. Repeat the "step" by passing from the second number to a third, and so on until stopped by meeting the first number in the way. The cycle of numbers thus found will have the "magic" sum. Thus, in A, $5+38+22+13+46+30+21=175$. This property may be called "step summation."

5. To produce a knight's path square which shall also be both symmetrical and pandiagonal, the root $r$ being any odd number not divisible by 3 or 5, say $2n+1$, we may begin in the middle, with the middle number, and write $n$ numbers succeeding, following the knight's path downward and to the right. When we have set down the $n^{\text{th}}$ number from the middle, that is, having made $n$ steps, this number being the first one reached which is divisible by $r$ [in A, 28], we must make a cross step before setting down the next number. This cross step may either be two places to our left and one down, or two to our right and one up, the general direction being at right angles to that of the previous steps, the knight's step always preserving its original character or "bend," viz. in any direction which it may take, two places forward and one to the left. [In A, the second of the two cross steps indicated is chosen; otherwise 29 would appear where 20 is.] Proceeding now again in the original direction, until further progress is obstructed [in A, 35 is such a stopping place, since the next step leads to the place occupied by 29], we find that a second cross step is necessary, similar to the one chosen for the first, and so on till the last number is written, when the cross step leads to the place where 1 must be written, after which the process is to be continued till the square is completed. [There is therefore a second

square having the properties of A, which the reader will, on laying out the 49 spaces, have no trouble in filling up off-hand.] In this manner we may produce two squares, possessed of this combination of properties, for each odd root not divisible by 3 or 5. Or, we may begin by placing 1 in the uppermost row, just to the left of the middle column, if the form of cross step "to our left" is chosen, and write the numbers directly in their order, making a cross step when obstructed, till the square is completed; or again, by placing 1 in the lowest row, just to the right of the middle column, and using the other form of cross step throughout. [See A.] A square of any size, however large, may be filled out in this way for the mere trouble of writing the numbers in their natural order. The rule has been given in both forms, starting from the middle and from 1 respectively, the former method giving more insight into the principles involved, the latter being derived from the former with a little assistance from algebra. Again, we may start from 1 in any position and produce either square in a non-symmetrical form, requiring one or two transpositions, as already explained, to bring the middle number to the middle.

6. What is here presented as a "knight's path square," meaning a square containing the numbers in their natural order arranged along a path consisting throughout of knight's steps having the same bend, is but a special case, one of a class of squares having similar properties, a class which may be referred to as "uniform step squares." The square B, wherein $r = 11$, is a second illustration. In A, the uniform step was "two places forward, one to the left," or left-bend knight's step. In B it is three places forward, whatever the direction may be,

B.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 54 | 7 | 81 | 34 | 119 | 72 | 25 | 110 | 63 | 16 |
| 85 | 38 | 112 | 76 | 29 | 103 | 56 | 20 | 94 | 47 | 11 |
| 69 | 33 | 107 | 60 | 13 | 98 | 51 | 4 | 78 | 42 | 116 |
| 64 | 17 | 91 | 55 | 8 | 82 | 35 | 120 | 73 | 26 | 100 |
| 48 | 1 | 86 | 39 | 113 | 77 | 30 | 104 | 57 | 21 | 95 |
| 43 | 117 | 70 | 23 | 108 | 61 | 14 | 99 | 52 | 5 | 79 |
| 27 | 101 | 65 | 18 | 92 | 45 | 9 | 83 | 36 | 121 | 74 |
| 22 | 96 | 49 | 2 | 87 | 40 | 114 | 67 | 31 | 105 | 58 |
| 6 | 80 | 44 | 118 | 71 | 24 | 109 | 62 | 15 | 89 | 53 |
| 111 | 75 | 28 | 102 | 66 | 19 | 93 | 46 | 10 | 84 | 37 |
| 106 | 59 | 12 | 97 | 50 | 3 | 88 | 41 | 115 | 68 | 32 |

then two to the left. The regular step, say from 61 in the middle to 62, or from 1 to 2, is three places down and two to our right, as we look at the square, our "right" being "left" to any one looking downward from the starting point. The cross step, say from 66 to 67, or from 11 to 12, is three places to the right and two up. A second square having the same properties may be produced by taking the opposite cross step, three places to the left and two down. If we count $x$ places to the right and $y$ places up, the regular step in B is defined by $x=2$, $y=-3$, and the cross step (let us say the cross step to our right) by $x=3$, $y=2$; and the cross step (to our left) for the other square would be defined by $x=-3$, $y=-2$. The general rule, when $r$ is prime and greater than 5, for producing a "uniform step square" is, first to take for the regular step $x=a$, $y=-b$, and for the cross step either $x=b$, $y=a$, or $x=-b$, $y=-a$, where $b$ is any number from 2 to $n$ inclusive, and $a$ is any number above 0 and less than $b$, provided that $a^2+b^2$ is prime to $r$. Here, as before, $n$ is such that $r=2n+1$. (Otherwise we may, subject to the same rare exceptions, define the choice of steps as from the middle place to any place in the south-southeast eighth of the square, using the points of the compass as in a map; any place, that is to say, included between the middle column and the diagonal, below and to the right of the middle place. This space may at pleasure be transferred to any other desired eighth of the square by turning the square around or using a real or imaginary looking-glass, so that the restriction of movement from the middle to a small fraction of the whole space does not impair the generality of the method. The same remark holds concerning the use of steps having only the left bend, which in the mirror would of course appear as having the opposite bend, such variations not affecting the identity of the square.) Having chosen the steps, we must now proceed exactly as explained for the knight's path square in the preceding paragraph, except that another place must be found for 1, if it be desired to begin with 1 instead of with the middle number. For the knight's step, $a=1$, $b=2$, and the place of beginning (with cross step to our right) is [see A] in the first row, counting from the bottom, and in the first column to the right of the middle column. When the cross step is to our left, the place of beginning is in all cases symmetrically, that is, centrally, opposite to the place of beginning when the cross step is to our right, so that it is unnecessary to discuss both cases. The latter case, "to our right," is presupposed in what follows. For all steps in which $a=1$, the place of beginning is in the first row, counting upward from the bottom; if $a = 3$, the second row; if $a = 5$, the third row, and so on. If $a = 2$, it is in the first row above the middle; if $a = 4$, the second, and so on. For all steps in which $b = 2$, the place of beginning is in the first column from the middle, counting to the right; if $b = 4$, the second, and so on. If $b = 3$, the place of beginning is in the second column, counting from the left; if $b = 5$, the third, and so on. Knowing these things, it is usually possible to begin the writing of the square by putting 1 in any designated place not in line with the middle place. For example, let it be demanded, in a square of 13 on a side, that 1 be placed in the fourth column, and in the second row from the top. Here $b = 6$, $a = 3$, and the cross step is to our left. Or, let the assigned place be in the first column, and in the second row from the top. This requires that the direction of the steps be changed. We may treat the first column as if it were the top row, and, thus giving the square an imaginary turn of 90°, we shall regard the row next to the top as if it were the column next to that on the right. The problem now is to place 1 in the top row and in the second column counting from the right. For the directions thus altered, we have $b = 3$, $a = 1$, cross step to the left. In reality, the regular step will be three places to the right, one upward, and the cross step will be three places down, one to the right.

7. For "uniform step squares," regarded more generally, in which the step is defined by coördinates $x$ and $y$, without restricting the direction of the step, we have for the cross steps the coördinates $x' = \mp y$, $y' = \pm x$, the upper sign referring to what has been spoken of as "the cross step to our right." Taking the middle place as the origin of coördinates, the location of the first multiple of $r$ which is reached in order, after taking $n$ steps from the middle number, is $nx$, $ny$, or rather the remainder of these after division by $r$; that of the number next higher, after making the cross step, is $nx \mp y$, $ny \pm x$, and that of 1, readily deduced from these, $x'' = \pm ny$, $y'' = \mp nx$, both expressions being likewise subject to reduction by any multiple of $r$, because as a coördinate $r = 0$. Thus, as before, if $x = a = 1$ and $y = -b = -2$, the location of 1 is $x'' = \mp 2n = \pm 1$, $y'' = \mp n$; that is, column $\pm 1$, row $\mp n$, as in the preceding paragraph. From these two general formulæ for the location of 1, namely, $x'' = \pm ny$, $y'' = \mp nx$, we learn that to produce a square after assigning 1 to column $x''$, row $y''$, measured from the middle of the square, we have merely to add as many times

$r$ respectively to $x''$ and $y''$ as shall make them each divisible by $n$, thus deriving $x$ and $y$ at once. Thus, if $r = 2n + 1 = 13$, $x'' = n - 1 = 5$, $y'' = -3$, the step for this location of 1 will be shown by $x = (y'' + 3r)/(\mp n) = \mp 6$, $y = (x'' + r)/(\pm n) = \pm 3$, as in paragraph 6. Since the signs of $x$ and $y$ when determined in this way depend on that of the cross step, two squares can always—subject to certain exceptions which will be noted—be developed from any location not in line with the middle which may be assigned to 1, one by the direct step, the other by the same step backwards taken with the opposite cross step. No step is available for producing a pair of "uniform step squares," in which the numbers follow the steps in their natural order, including of course under this name the class of knight's path squares first described, unless $r$ is prime to both $x$ and $y$, and also prime to both the sum and the difference of the natural numbers indicated by those letters, as well as to the sum of their squares.

8. Some attention may for a moment be drawn to those failing cases wherein $r$ is not prime to $a^2 + b^2$. It is not practically necessary to speak of composite values of $r$, the lowest of which, under the restrictions stated, is 49. Let us glance at the successive prime numbers 7, 11, 13, 17, 19, 23. For every value of $r$, if there were no failing cases, there would be $\frac{1}{8}(r - 1)(r - 3)$ steps, each producing two squares with opposite cross steps. There are 3 such steps for $r = 7$, namely, $a = 1$, $-b = 2$ or $3$; $a = 2$, $-b = 3$. There is likewise no failing case for $r = 11$, the steps being 10 in number, namely, $a = 1$, $-b = 2, 3, 4$, or $5$; $a = 2$, $-b = 3, 4$, or $5$; $a = 3$, $-b = 4$ or $5$, $a = 4$, $-b = 5$. For $r = 13$, the failing cases are $a = 1$, $-b = 5$; $a = 2$, $-b = 3$; $a = 4$, $-b = 6$; leaving 12 available steps. For $r = 17$, the failing cases are $a = 1$, $-b = 4$; $a = 2$, $-b = 8$; $a = 3$, $-b = 5$; $a = 6$, $-b = 7$; leaving 24 available steps. For $r = 19$ and $r = 23$ there are respectively 36 and 55 available steps without failing cases.

9. The most important novel element in the knight's path method, and in the more general uniform step method of which the knight's path method is a special case, consists in the exhibition of uniform steps by which the numbers are written down throughout in order, perhaps starting offhand with 1 in a place arbitrarily assigned. If for any purpose it be desired to follow the same series of steps while employing a series of numbers not in their natural order, it is possible to do so, and still to produce a square both symmetrical and pandiago-

nal, the path pursued, however, being no longer evident after completion of the process. The way of doing this will be shown most readily by an example, in which $r = 7$, the same process being obviously applicable when $r$ has other prime values. Let the numbers from 1 to 7 be arranged in any order, subject to certain conditions. The middle number, 4, must not be changed. The two numbers next to it on either side must have 8, that is, $3 + 5$, the same as before, for their sum. The two numbers next adjacent must have the same sum, and so on. Thus we may perhaps reach some such order as this: 3, 7, 2, 4, 6, 1, 5. Call this series $S_0$. Form another series, $S_1$, by adding 7 to each term of $S_0$; then another, $S_2$, by adding 14 to each term of $S_0$, and so on till 7 series, ending with $S_6$, have been written down. Now rearrange the letters $S_0$, $S_1$, .... $S_6$, retaining $S_3$ in the middle, in any order, provided the sum of the subscripts of any two equidistant from $S_3$ shall remain 6 as before, and suppose the result is $S_5$, $S_0$, $S_4$, $S_3$, $S_2$, $S_6$, $S_1$. If for these letters we substitute the numbers which they represent, we shall have as the result the numbers from 1 to 49 arranged thus: 38, 42, 37, 39, 41, 36, 40, 3, 7, 2, 4, 6, 1, 5, 31, 35, 30, 32, 34, 29, 33, 24, 28, 23, 25, 27, 22, 26, 17, 21, 16, 18, 20, 15, 19, 45, 49, 44, 46, 48, 43, 47, 10, 14, 9, 11, 13, 8, 12. If of the three possible steps for $r = 7$ we

C.

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 41 | 12 | 49 | 18 | 22 | 31 |
| 43 | 17 | 23 | 34 | 5 | 42 | 11 |
| 35 | 4 | 36 | 10 | 44 | 20 | 26 |
| 13 | 47 | 21 | 25 | 29 | 3 | 37 |
| 24 | 30 | 6 | 40 | 14 | 46 | 15 |
| 39 | 8 | 45 | 16 | 27 | 33 | 7 |
| 19 | 28 | 32 | 1 | 38 | 9 | 48 |

choose the knight's path, and of the two cross steps choose the one to our right, we shall follow the order of the steps shown in A, but by using the numbers in their new order we shall produce the square C, which is both symmetrical and pandiagonal, but in which the knight's path is not obtrusive to the eye. Some other uniform step, however, less known than the knight's path, should be chosen if complete disguise is desired.

10. It is not difficult to prove that the method of uniform steps, for which one of the steps which may be chosen is the knight's step, must produce squares

both symmetrical and pandiagonal, provided the numbers are written in what may be called "symmetrical order," that is to say, either in their natural order, or rearranged symmetrically in $r$ series of $r$ numbers each as prescribed in the foregoing paragraph. Any square so formed must be symmetrical, because the middle number of the middle series is the middle number of all and is set in the middle place of the square; the two numbers next it either way have the uniform sum $r^2 + 1$, one of them being located by a step forward, the other by a step backward, so as to occupy places symmetrically opposite each other; the next pair of numbers are similarly opposite each other, and so on by pairs throughout.

11. Any square so formed must be pandiagonal, because it satisfies La Hire's requirement for what he called "perfect" squares. He divided the numbers into $r$ series of $r$ numbers each, without reference to symmetry, regard for which appears always to have been slighted, and nearly always unthought of, by writers on pandiagonal squares. His system of dividing the numbers into series, which has been fruitful in the hands of subsequent writers, was to regard every number as the sum of two constituents, say an elementary number $p$ such as 1, 2, .... $(r-1)$, $r$, and a base number $q$ such as 0, $r$, $2r$, .... $(r-1)r$. The elementary numbers, after being arranged in any order, may be designated in their new order as $p_1$, $p_2$, .... $p_r$, and the base numbers, similarly arranged in any order, as $q_0$, $q_1$, $q_2$ .... $q_{r-1}$. Any one of the original numbers, from 1 to $r^2$, is known now as the sum of its two constituents, say $q_k + p_m$. La Hire observed that if a square were first formed of $q$'s, each $q$ being repeated $r$ times, in such a manner that the same $q$ did not appear twice in the same line, that is, the same row, column, or diagonal, whole or broken, the square would be pandiagonal; and that if another square were similarly formed of $p$'s, each $p$ being repeated $r$ times, but arranged in some different order from that followed in the $q$ square, this also would be pandiagonal; and that the two squares might be superimposed, the constituents falling together being added so as to produce a "perfect" square. It is enough for us if the same series (see paragraph 9) is not represented twice in the same line, and that two numbers of the same rank in different series do not appear together in the same line.

12. That the same series is not represented twice in the same row is plain, because each series of $r$ numbers is located by $r$ steps of uniform character, each

measuring $b$ places down, $a$ places to the right, and $b$ is taken prime to $r$. It is not represented twice in the same column, similarly, because $a$ is prime to $r$. And it is not represented twice in the same diagonal, because each step leads to the $(b+a)^{\text{th}}$ diagonal whose direction is downward to the left and to the $(b-a)^{\text{th}}$ diagonal whose direction is downward to the right, and both $b+a$ and $b-a$ are prime to $r$. Again, two numbers of the same rank in different series cannot appear in line together unless the two leading numbers of those series are in line together, for the several series march in, so to speak, parallel order with equal steps. We must therefore examine the steps by which the leading numbers of the several series follow one upon the other, steps which from the nature of the whole network must be uniform. It is sufficient to consider only the cross step to our right, the same reasoning sufficing for the case in which the other cross step is chosen. The regular step from 1 to 2, assuming for brevity that the numbers are in their natural order, is $y=-b$, $x=a$. The step backward, therefore, from 1 to $r$, is $y=b$, $x=-a$. The cross step from $r$ to $r+1$ is $y=a$, $x=b$. The step from 1 to $r+1$ is therefore $y=b+a$, $x=b-a$. Since these numbers and their sums and differences are prime to $r$, no two of the leading terms $1, r+1, 2r+1, \ldots\ldots,$ can be in line together; and it follows, as stated, that no two numbers of the same rank in different series can be in line together, so that any square produced by the method of uniform steps is pandiagonal.

13. The method must obviously produce a square whenever the step chosen is such as not to interfere with itself, so to speak, an interference which must happen whenever a succession, less than $r$ in number, of cross steps, each defined by $y=a$, $x=b$, leads to a place which is likewise to be reached from the same starting point by a succession of less than $r$ regular steps, each defined by $y=-b$, $x=a$. Let us suppose such a place to be reached by $p$ regular steps, and by $q$ cross steps. Its location, taking the starting point as the origin, is $y=-pb=qa$, and $x=pa=qb$, each expression when greater than $r$ being reducible by the subtraction of the arithmetical value of $r$ or some multiple of it. Thus, assigning due values to $j$ and $k$, we have $jr-pb=qa$, and $qb=pa+kr$. From these by multiplying we derive $jbr-pb^2=pa^2+kar$, whence $a^2+b^2=mr$, where $m=(jb-ka)/p$, and $p$ is less than $r$. Interference is therefore avoided (see paragraph 8) when $a^2+b^2$ is prime to $r$. Sup-

pose, for an example of interference, $r = 13$, $a = 1$, $b = 5$. Here $m = 2$, and the simplest values suitable are $p = 5$, $q = 1$, $j = 2$, $k = 0$; the first cross step clashes with the fifth direct. Again, suppose an attempt made to write a knight's path square with $r = 25$, $a = 1$, $b = 2$. Here $m = 1/5$, according with $p = 10$, $q = 5$, $j = 1$, $k = 0$; the fifth cross step clashes with the tenth direct. When $r$ is prime it is sufficient to assume $q = 1$, $k = 0$.

14. In addition to uniform step squares, other forms may be produced which shall be both symmetrical and pandiagonal, proceeding as before by a regular step for the first series of $r$ numbers, arranged symmetrically as in paragraph 9 if not in their natural order, but using a different cross step. If for brevity we call the first of the first series 1 and the last of the same series $r$, the rule, when $r$ is prime and $> 3$, may be laid down that the cross step from $r$ to $r + 1$, the first of the next series, may be taken to any place not in line with 1 and not already occupied by a number of the first series. To prove this it is only necessary to show that the steps 1, 2, 3, .... and 1, $r + 1$, $2r + 1$, .... cannot lead to a common place of meeting before their return to the place of beginning where 1 is located. If one series of steps be denoted by $x$, $y$, the first place will be located by $x$, $y$; the second by $2x$, $2y$; the third by $3x$, $3y$; and so on to $rx$, $ry$, which is the same as 0, 0, the place of beginning. Let us suppose a second series of steps, each denoted by $2x$, $2y$; these will reach the same places, $r$ in number, though in different order, namely, $2x$, $2y$; $4x$, $4y$; .... $(r-1)x$, $(r-1)y$; $(r+1)x$, $(r+1)y$, that is, $x$, $y$; then $3x$, $3y$; .... $rx$, $ry$, as before; and this must hold good whether $2x$ or $2y$, if greater than $r$, is counted in full or reduced by $r$, since multiples of $r$ will not affect locations on the square. In the same way we shall see that a step from 1 to any one of the places in question, repeated $r$ times, must reach each other of the same places and no other place. The like is true of the other series of steps, leading successively to the series of places of 1, $r + 1$, $2r + 1$, .... Since $r + 1$ was assigned to a place not occupied by any one of the numbers 1, 2, .... $r$, the two paths are therefore wholly distinct. The work may either be begun with the middle number in the middle place, or a square "capable of symmetry" may be produced by beginning with 1 in any position. No discussion is here contemplated concerning the formation of squares of this irregular sort when $r$ is not prime. It will be remarked that this variation of the method must be used when $r = 5$, whenever

a square both symmetrical and pandiagonal is desired, since in this case $r$ cannot be prime to $a^2+b^2$, so that uniformity of step throughout is not possible.

15. The reasoning of the preceding paragraph may also manifestly be applied to cases where the cross step is uniform with the direct step. It is to be remarked that it is not possible to take a cross step which shall have a different "bend" while uniform with the direct step in other respects, because such a step would lead to a place for $r+1$ in line with the place of 1, which is forbidden. The direct step being $x=a$, $y=-b$, the position of $r$, measured backward from 1, is $x=-a$, $y=b$. The cross step with the other bend from $r$ to $r+1$ being $x=\mp b$, $y=\pm a$, the position of $r+1$, measured from 1, is found to be $x=-(a\pm b)$, $y=\pm(a\pm b)$, showing that $r+1$ and 1 are in line diagonally with each other. This explains, for example, why no attempt has here been made to produce a knight's path square with a cross step having a different bend from that of the direct step. It is almost unnecessary to observe that if $r+1$ were in line with 1 the line could not have the magic sum $\frac{1}{2}r(r^2+1)$, since its sum would be $1+r+1+2r+1+\ldots+(r-1)r+1=\frac{1}{2}r(r^2-r+2)$. When a different cross step is used, as in paragraph 14, it is not impossible to produce squares both symmetrical and pandiagonal for odd values of $r$, such as 15 or 25, which do not permit the off-hand formation of uniform step squares.

## PART II.—*The Figure-of-Eight Method.*

16. Symmetry, when the root $r$ is even, is less useful a quality than when the root is odd, as there is no middle place from which to measure distances. The pandiagonal quality is still essential, when $r$ is divisible by 4. It is not feasible for other even values of $r$. Let us assume without further repetition that $r$ is divisible by 4. Pandiagonal squares of the best form for such values of $r$—let us call them "complete" squares—possess the following combined properties: first, they possess all their properties without diminution however much the rows and columns may be transposed (see paragraph 3), differing in this respect from symmetrical pandiagonal squares for odd values of $r$; second, they possess additional magic summations by blocks of four, any small square of four being chosen as a block, and enough blocks being chosen, overlapping or otherwise, to make up $r$ numbers in all; third, each number is complementary to the one distant from it $\frac{1}{2}r$ places in the same diagonal. The second property pro-

D.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 63 | 3 | 61 | 12 | 54 | 10 | 56 |
| 16 | 50 | 14 | 52 | 5 | 59 | 7 | 57 |
| 17 | 47 | 19 | 45 | 28 | 38 | 26 | 40 |
| 32 | 34 | 30 | 36 | 21 | 43 | 23 | 41 |
| 53 | 11 | 55 | 9 | 64 | 2 | 62 | 4 |
| 60 | 6 | 58 | 8 | 49 | 15 | 51 | 13 |
| 37 | 27 | 39 | 25 | 48 | 18 | 46 | 20 |
| 44 | 22 | 42 | 24 | 33 | 31 | 35 | 29 |

duces a fourth, that of alternate equivalent couplets. For example, the square D is one in which $r = 8$; in which every block of four has the sum 130, so that any two blocks have the magic sum 260; and in which every number and its diagonal fourth have the sum 65. The sum of any two overlapping blocks being equal, it follows that all alternate couplets have equal sums. Thus $1 + 16 = 3 + 14$, $50 + 47 = 52 + 45$, $63 + 3 = 47 + 19$, and so on throughout without exception, both vertically and horizontally. A fifth property is an easy consequence of the fourth. The alternate couplets being equivalent, the four corners of any rectangle whatever, having an even number of places on each side, constitute a block again possessed of half the magic sum, so that any $\frac{1}{4}r$ such blocks, however different in size or shape, whether apart or overlapping, will have the magic sum. The magic and pandiagonal properties themselves follow necessarily in these squares from the third and fourth: as regards the whole and broken diagonals, directly from the third, or perhaps rather from a sixth property which is a corollary of the third, namely, that any selected $\frac{1}{2}r$ numbers in the square added to the $\frac{1}{2}r$ numbers complementary to them in the same diagonals respectively, distant each from its complement $\frac{1}{2}r$ places, will have the magic sum. Of each row or column, one-half is composed of the complements of couplets which are alternate with and equivalent to the couplets composing the other half, so that the row or column again has the magic sum. What is obviously to be desired is a simple method of producing squares possessed of the second and third properties, from which all the others are thus seen to follow. The problem is in fact to distribute $\frac{1}{2}r^2$ non-complementary numbers in $\frac{1}{2}r$ adjacent rows or columns, forming one-half of the square, so as to exhibit the second or "blocks of four" property throughout the whole square when it is completed by adding the complementary numbers.

17. The square D was devised by what may be called the figure-of-eight method, because the order in which the rows are first written bears some resemblance to the figure 8 laid on one side, the usual sign for infinity. The upper half of the square was first filled as indicated below. By following the

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 12 | 11 | 10 | 9 |
| 16 | 15 | 14 | 13 | 5 | 6 | 7 | 8 |
| 17 | 18 | 19 | 20 | 28 | 27 | 26 | 25 |
| 32 | 31 | 30 | 29 | 21 | 22 | 23 | 24 |

order of the numbers from 1 to 32, the reason for using the phrase in question will readily be seen. The numbers in every alternate column, second, fourth, etc., were then replaced by their complements, and this supplied the upper half of D, the lower half being added by writing in the complements as indicated in the last paragraph. The rule therefore for producing "complete" squares is to write the first $\frac{1}{2}r$ numbers in the first row, then drop to the second row, returning backwards along the first row and dropping to the second so as to complete both rows in what we may call the figure-of-eight manner. The next two rows must come next in the same way, and so on till half the square is filled, when every alternate column is to be replaced by the complementary numbers, after which the rest of the square is to be completed by writing down the complement of each number in the same diagonal, $\frac{1}{2}r$ places lower down. The numbers may be arranged in their natural order, or in an appropriate artificial order, as will be seen later, but no other variation is proposed.

18. Let us see what happens upon taking the odd numbers first, as seen below. If we replace the first, third, fifth, and seventh columns by writing in

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 23 | 21 | 19 | 17 |
| 31 | 29 | 27 | 25 | 9 | 11 | 13 | 15 |
| 33 | 35 | 37 | 39 | 55 | 53 | 51 | 49 |
| 63 | 61 | 59 | 57 | 41 | 43 | 45 | 47 |

lieu of them the complements of the numbers composing them, and supply the lower half, we obtain the "complete" square E. We might begin also by

E.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 64 | 3 | 60 | 7 | 42 | 21 | 46 | 17 |
| 34 | 29 | 38 | 25 | 56 | 11 | 52 | 15 |
| 32 | 35 | 28 | 39 | 10 | 53 | 14 | 49 |
| 2 | 61 | 6 | 57 | 24 | 43 | 20 | 47 |
| 23 | 44 | 19 | 48 | 1 | 62 | 5 | 58 |
| 9 | 54 | 13 | 50 | 31 | 36 | 27 | 40 |
| 55 | 12 | 51 | 16 | 33 | 30 | 37 | 26 |
| 41 | 22 | 45 | 18 | 63 | 4 | 59 | 8 |

writing the even numbers, but the result would be the same square, upside down, written backwards, and transposed. It is also to be remarked generally that it makes no real difference which set of columns is selected for replacement, whether the first, third, etc., or the second, fourth, etc. If, for example, the other set of columns had been replaced in this case by the complementary numbers, the resulting square would have been what E becomes after such transpositions as are required to bring 1 to the upper corner on the left hand. The reader will find on trial that the numbers may also be taken at intervals of 4, viz. 1, 5, 9, .... 61, followed by 2, 6, 10, .... 62. Illustrations of like results for larger squares, as where $r = 12$, $r = 16$, etc., may be multiplied to any extent.

19. Since other ways of arranging the numbers in order are doubtless available, while certainly the numbers cannot be arranged at random, it becomes necessary to examine the principle underlying this method, so as to ascertain the limits within which the order of the numbers can be changed. Let the first $r$ numbers in the required artificial order be $a_1, a_2, \ldots . a_r$; the second $r$ numbers $b_1, b_2, \ldots . b_r$, and so on. Let the sum of any vertical couplet of the first and second row, as first arranged, be $s_1$; of the second and third row, $s_2$, and so on. This is then the first arrangement:

$$\begin{array}{llllllll}
a_1 & a_2 & \ldots\ldots\ldots\ldots & a_{\frac{1}{2}r} & b_{\frac{1}{2}r} & \ldots\ldots\ldots\ldots & b_2 & b_1 \\
b_r & b_{r-1} & \ldots\ldots\ldots\ldots & b_{\frac{1}{2}r+1} & a_{\frac{1}{2}r+1} & \ldots\ldots\ldots\ldots & a_{r-1} & a_r \\
c_1 & c_2 & \ldots\ldots\ldots\ldots & c_{\frac{1}{2}r} & d_{\frac{1}{2}r} & \ldots\ldots\ldots\ldots & d_2 & d_1 \\
d_r & d_{r-1} & \ldots\ldots\ldots\ldots & d_{\frac{1}{2}r+1} & c_{\frac{1}{2}r+1} & \ldots\ldots\ldots\ldots & c_{r-1} & c_r \\
\ldots & \ldots & & \ldots\ldots & \ldots\ldots & & \ldots\ldots & \ldots \\
m_r & m_{r-1} & \ldots\ldots\ldots\ldots & m_{\frac{1}{2}r+1} & l_{\frac{1}{2}r+1} & \ldots\ldots\ldots\ldots & l_{r-1} & l_r .
\end{array}$$

The relations which are required for our purpose are: $a_x + b_{r+1-x} = s_1$, $b_x + c_{r+1-x} = s_2, \ldots . \; l_x + m_{r+1-x} = s_l$, and also, as will be shown immediately, $m_{\frac{1}{2}r+x} - b_x = l_{\frac{1}{2}r+x} - a_x = t$, another constant sum, positive or negative. Also, no two of the numbers in this scheme, representing the first arrangement of the upper half of the proposed square, can be complementary, that is, their sum must not be $r^2 + 1$. Let us represent $r^2 + 1$ by $\rho$. It is immaterial whether the alternate columns be replaced by the complementary numbers before or after the lower half of the square is filled out complementarily; for the moment, we may assume the lower half filled first. The two expressions here given for $t$ correspond to those preceding them, to this extent, that if we denote by $s_m$ the constant sum of numbers in the row beginning with $m_r$ and those respectively below them in the first complementary row, viz. $\rho - b_{\frac{1}{2}r}, \ldots .$, we shall have $m_{\frac{1}{2}r+x} + \rho - b_x = s_m$, so that $t$ represents $s_m - \rho$. The foregoing relations are sufficient, because when they exist any two adjacent vertical couplets must have the same sum, say $k$, and when one of these two couplets is replaced by its complementary couplet, the sum of which is $2\rho - k$, the block of four thus formed has the required sum $2\rho$. Since any two adjacent rows have constant sums, each row and the second, or fourth, etc., row from it must have constant differences, so that $l_x - a_x = m_x - b_x$ is constant for values of $x$ from $\frac{1}{2}r + 1$ to $r$ inclusive; or let us say that $l_{\frac{1}{2}r+x} - a_{\frac{1}{2}r+x} = m_{\frac{1}{2}r+x} - b_{\frac{1}{2}r+x} = g$, a constant. Then, $g = t + a_x - a_{\frac{1}{2}r+x} = t + b_x - b_{\frac{1}{2}r+x}$. From this, if we take $t - g = u_1$, we have this special relation, $a_{\frac{1}{2}r+x} = u_1 + a_x$, $b_{\frac{1}{2}r+x} = u_1 + b_x$. For the next two rows, similarly, $c_{\frac{1}{2}r+x} = u_2 + c_x$, $d_{\frac{1}{2}r+x} = u_2 + d_x$, and so on for every pair of rows. Conversely, if the rows are thus arranged, the final relation first stated, containing $t$, and involving the first complementary row, will follow. We may therefore choose $\frac{1}{2}r$ numbers, $a_1, a_2, \ldots . a_{\frac{1}{2}r}$, and by adding $u_1$ to each derive successively $a_{\frac{1}{2}r+1}$ to $a_r$; then $b_1 = s - a_r$, $b_2 = s - a_{r-1}, \ldots .$; then $c_1 = a_1 + p$, $c_2 = a_2 + p$, $\ldots . c_{\frac{1}{2}r} = a_{\frac{1}{2}r} + p$; then $c_{\frac{1}{2}r+1} = c_1 + u_2, \ldots .$; then $d_1 = b_1 + p$ and so on. For $e_1$ we must introduce another constant, say $q$, such that $e_1 = a_1 + q$; and for $e_{\frac{1}{2}r+1}$ another, say $u_3$; and so on. Thus, if $a$ represent any one of the original $\frac{1}{2}r$ numbers chosen, the others will follow as in the schedule marked F, $b$ being derived by subtracting the proper $a$ from $s$. The choice of the original numbers, $a_1$ to $a_{\frac{1}{2}r}$, is restricted by the requirement that they, with all the numbers derived from them by the assignment of $s$ and of the differences $p, q, \ldots ., u_1, u_2, \ldots .,$

F.

| | $x=1$ to $x=\frac{1}{2}r$. | $x=\frac{1}{2}r$ to $x=r$. |
|---|---|---|
| $a$ | $a$ | $a+u_1$, |
| $c$ | $a+p$ | $a+p+u_2$ |
| $d$ | $b+p$ | $b+p+u_2$ |
| $e$ | $a+q$ | $a+q+u_3$ |
| $f$ | $b+q$ | $b+q+u_3$ |

whether such differences be positive or negative, making $\frac{1}{2}r^2$ numbers in all, shall be non-complementary throughout.

| | | | |
|---|---|---|---|
| $a_1$ | 1 | 1 | 1 |
| $a_2$ | 2 | 2 | 2 |
| $s$ | 9 | 13 | 9 |
| $u$ | 1 | 1 | 4 |

20. The simplest illustration is derived, of course, from the case $r=4$. The three available sets of values corresponding to the three pandiagonal squares of 4, all of which are necessarily "complete," are shown in the margin. Other values of $a_1$, etc., reproduce the same squares. It would involve much study to determine the number of possible complete squares of 8 and assign the values corresponding. In the simple case D, where the numbers are taken in natural order, we have $s=2r+1=17$, $u_1=u_2=\frac{1}{2}r=4$, $p=2r=16$; and in general, for the natural order in all cases where $r=4n$, we have $u_1=u_2=\ldots\ldots=2n$, $p=q=\ldots\ldots=2r$, $s=2r+1$. An obvious variation is obtained by changing the order of the numbers $a_1, a_2, \ldots\ldots a_{\frac{1}{4}r}$, while retaining the same values of $s$, $u_1$, etc., and this sort of variation is available for every complete square, however obtained. The result is to interchange the columns in like order, prior to the complementary substitution. In any complete square the odd-numbered columns, first, third, etc., of the left half may therefore be interchanged in any order, provided those of the right half are interchanged in like order, and the like is true of the even-numbered columns among themselves. By turning the square, columns become rows, so that the like is true of rows. It is easy to show algebraically that for squares turned partly around, or written backwards, or both, the numbers in their new relative positions are subject to the same rules of formation. For example, if the square D be so turned that the top row reads 44, 37, 60, ...., we can have $a_1=44$, $a_2=65-37=28$, $a_3=60$, $a_4=12$, $s=66$, $u_1=-10$, $p=-2$, $u_2=-6$.

21. No square can be "complete" which cannot be analyzed according to this method of formation, as shown in F. Each block of four, which must have the sum $2\rho$, is composed of two pairs of numbers, or couplets, whose sums respectively are, let us say, $\alpha$ and $\beta$. If the second couplet is assumed to have been derived from an earlier one by complementary substitution, the sum of the earlier couplet is $2\rho - \beta = \alpha$, so that any two adjacent couplets must, prior to the substitution, have had the same sum $\alpha$. This, together with the stated properties of the complete square, is all that was presupposed in paragraph 19.

22. There are many ways more or less formal, of arranging the order of the numbers in applying the general method, besides those simplest ways mentioned in paragraphs 17 and 18. It will often be convenient to begin by selecting $2r$ numbers in arithmetical progression such that if to each be added $p$, positive or negative, for $r = 8$; $p$ and $q$ separately, for $r = 12$; three such constants, for $r = 16$, and so on; the numbers so found, including the original $2r$ numbers, making $\frac{1}{2}r^2$ in all, shall all be different and non-complementary. The $2r$ numbers so taken may then be arranged in $r$ pairs having a uniform sum $s$, the largest being paired with the smallest, and so on. Then, taking only one number from a pair, it is necessary to choose $\frac{1}{2}r$ numbers for $a_1, a_2$, etc., such that the other $\frac{1}{2}r$ shall severally differ from them respectively by a constant difference $u_1$. As an illustration, let $r = 12$, and let us choose the first 24 for our $2r$ numbers, in pairs, having $s = 25$, viz. 1 – 24, 2 – 23, 3 – 22, 4 – 21, 5 – 20, 6 – 19, 7 – 18, 8 – 17, 9 – 16, 10 – 15, 11 – 14, 12 – 13. We may elect to take for the first six $a$'s 1, 2, 3, 7, 8, 9, since by adding 12 to each, say $u_1 = 12$, we reach the other six pairs as required, giving the other six $a$'s. If now for the utmost simplicity we choose $p = q = 24$, $u_2 = u_3 = 12$, the numbers throughout assume this order, when arranged in figures-of-eight:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 7 | 8 | 9 | 12 | 11 | 10 | 6 | 5 | 4 |
| 24 | 23 | 22 | 18 | 17 | 16 | 13 | 14 | 15 | 19 | 20 | 21 |
| 25 | 26 | 27 | 31 | 32 | 33 | 36 | 35 | 34 | 30 | 29 | 28 |
| 48 | 47 | 46 | 42 | 41 | 40 | 37 | 38 | 39 | 43 | 44 | 45 |
| 49 | 50 | 51 | 55 | 56 | 57 | 60 | 59 | 58 | 54 | 53 | 52 |
| 72 | 71 | 70 | 66 | 65 | 64 | 61 | 62 | 63 | 67 | 68 | 69 |

## PART III.—*Previous Approaches to these Methods.*

23. The annexed squares G and H were produced some four centuries ago

G.

| | | | | | | |
|---|---|---|---|---|---|---|
| 22 | 47 | 16 | 41 | 10 | 35 | 4 |
| 5 | 23 | 48 | 17 | 42 | 11 | 29 |
| 30 | 6 | 24 | 49 | 18 | 36 | 12 |
| 13 | 31 | 7 | 25 | 43 | 19 | 37 |
| 38 | 14 | 32 | 1 | 26 | 44 | 20 |
| 21 | 39 | 8 | 33 | 2 | 27 | 45 |
| 46 | 15 | 40 | 9 | 34 | 3 | 28 |

H.

| | | | | | | |
|---|---|---|---|---|---|---|
| 38 | 14 | 32 | 1 | 26 | 44 | 20 |
| 5 | 23 | 48 | 17 | 42 | 11 | 29 |
| 21 | 39 | 8 | 33 | 2 | 27 | 45 |
| 30 | 6 | 24 | 49 | 18 | 36 | 12 |
| 46 | 15 | 40 | 9 | 34 | 3 | 28 |
| 13 | 31 | 7 | 25 | 43 | 19 | 37 |
| 22 | 47 | 16 | 41 | 10 | 35 | 4 |

H.

| | | | | | | |
|---|---|---|---|---|---|---|
| λη | ιδ | λβ | α | ϰ6 | μδ | ϰ |
| ε | ϰγ | μη | ιζ | μβ | ια | ϰθ |
| ϰα | λθ | η | λγ | β | ϰζ | με |
| λ | 6 | ϰδ | μθ | ιη | λ6 | ιβ |
| μ6 | ιε | μ | θ | λδ | γ | ϰη |
| ιγ | λα | ζ | ϰε | μγ | ιθ | λζ |
| ϰβ | μζ | ι6 | μα | ι | λε | δ |

by Moschopulus of Constantinople.* His original Greek form of H is given here as a matter of interest. For the printer's convenience the cursive digamma representing 6 is replaced by 6. It will be seen that G is symmetrical and that H is pandiagonal and "capable of symmetry." Only a single author, so far as the writer's knowledge extends,† has noticed that certain pandiagonal squares are capable of symmetry; and it is most remarkable, for example, that the possibility of producing from H a square both pandiagonal and symmetrical by removing the two upper rows to the bottom should have escaped, if indeed it has escaped, the attention of the many acute computers, including a number of excellent mathematicians, who have dealt with this subject. The author referred to is the Rev. A. H. Frost,‡ who rediscovered the second rule of Moschopulus, unaware of its history, and indeed reproduced H in a varied form, and announced that squares derived by that rule could be made symmetrical. His object, however, was to produce pandiagonal squares, and in speaking of symmetry he referred only to the location of complementary numbers in opposite places.||

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*See Günther, "Vermischte Untersuchungen," Leipzig, 1876, for many historical details concerning magic squares, including a reprint of the essay of Moschopulus. The squares G and H had already been reprinted, in our notation, with an account of the methods of Moschopulus, by Mollweide.

† This saving clause, which for convenience will be suppressed in what follows, will kindly be understood and supplied by the reader concerning every other historical statement herein contained. It is needed, for very many have written on this subject in all sorts of odd ways and places.

‡ Quarterly Journal, XV, 43, dated by author February, 1877.

|| See paragraph 2, *ante*. A previous paper by Mr. Frost will be mentioned later.

24. The second rule of Moschopulus, illustrated by H, is merely that special case of the method of paragraph 14 wherein the direct step is the knight's, two places down and one to the right, and the cross step four places down. Another rule involving a cross step in our sense, that is, for example, when $r = 7$, the step from 7 to 8 and not the step from 1 to 8, was given by Mr. Frost in an earlier paper,* in which, with the knight's step down as the direct step, he prescribed one place up for what is here called the cross step, and remarked that squares so produced are capable of symmetry. Cross steps were treated freely by the late President Barnard of Columbia College,† who prescribed analytic tests by which to learn whether any given cross step is permissible in connection with any given direct step. The simple criterion of paragraph 14 could not have occurred to him; and in fact in his discussion of cross steps he had always in view the transition from 1 to 8—using the same example—rather than that from 7 to 8. It is perhaps owing to so many writers having followed La Hire in attending to the relation between 1 and 8, and to so few having followed Moschopulus in attending to that between 7 and 8, that the idea of a cross step uniform with the direct step has not heretofore been brought forward. Like other writers, Barnard appears not to have thought of the possibility of producing odd squares both symmetrical and pandiagonal.

25. Apart from Frost, the only writer known to have produced, even casually, in isolated cases, odd squares at once symmetrical and pandiagonal is Frolow,‡ who showed that the symmetrical square G becomes pandiagonal if its rows be written in the order 6, 3, 7, 4, 1, 5, 2, making a new square which we see to be the same as H with the two upper rows written below the others; and that a single square formed just like G for each other prime value of $r$ above 3 becomes likewise pandiagonal by a similar commutation of rows. Like Frost, Frolow failed to extend the notion of symmetry beyond the bare remark that two complementary numbers lie opposite each other throughout.

26. The property of "step summation" explained in paragraph 4 was described fully by Barnard as pertaining to all "perfect," here called pandiag-

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* Quarterly Journal, VII, 97, dated by author August, 1864.

† Johnson's Cyclopædia, first edition, article "Magic Squares"; an able and comprehensive treatise. The preface of this volume bears date August, 1876; the title, 1877.

‡ "Le Problème d'Euler," St. Petersburg, 1884.

onal, squares. From his subsequently referring to the later part of his treatise as original, it may be presumed that this point was drawn from some earlier author. It was independently discovered and published in 1877 by Frost, and again published by him later in the article "Magic Squares" of the Encyclopædia Britannica. Unaware of its earlier discovery, he gave it the name of "nasical summation," from the village of Nasik in India, where he resided when first engaged upon this subject.

27. No uniform step square, whether by the knight's step or any other, appears to have been produced heretofore having the pandiagonal property. A symmetrical knight's path square was devised by Euler, for the case $r = 5$. It may be found in the Encyclopædia Britannica.

28. The proof in paragraph 14 of the independence of two paths is identical in principle with that suggested by Frost for the independence of the various "normal paths" existing within a square of given dimension. The criterion of paragraph 14, thus proved, may also be drawn from the analytic data of Barnard's treatise, which would have aided materially any one to whom it had occurred to experiment with uniform steps, though it gives no hint towards originating that idea.*

29. The "blocks of four" property, for $r > 4$, seems due to Benjamin Franklin, whose square of 16 is reprinted by Günther. It is not pandiagonal, nor does it follow any uniform law, but it is so ingeniously put together that every "block of four" without exception has the uniform sum 514. Three errors, obviously typographical, require correction.

30. The "blocks of four" property is to be found in an incomplete form in many known pandiagonal squares, that is to say, it holds good for many blocks of the square, but not universally. In one at least, namely, the "magic square of squares" set forth by Barnard, both properties are universal, but the square

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*This idea, in point of fact, occurred to the writer when he was examining, on page 209 of Günther's treatise, a square of 18 produced by the second rule of Moschopulus. He had, while unacquainted with the papers of Barnard and Frost, been working up all possible symmetrical pandiagonal squares of 5 by diophantine methods. A chance observation that the published square of 18 was what is here called "capable of symmetry" led to an investigation of this method of Moschopulus, and to its extension as now shown.

fails in what is here stated as the first property of a complete square, that any row or column or any number of rows or columns may be transposed without changing in any manner the properties of the square. The special square referred to is not arranged complementarily in the manner defined here to be essential.

31. But one complete square appears to have been published heretofore, and that unconsciously, the existence of the "blocks of four" property not being mentioned. It was given as a pandiagonal square by Frost in his paper of 1877. Determining its elements according to the present method, they appear, in their simplest form, to be: $r = 8$, $a_1 = 1$, $a_2 = 7$, $a_3 = 3$, $a_4 = 5$, $s = 17$, $u_1 = 8$, $p = 16$, $u_2 = 8$.

32. Writers on magic squares have always recognized peculiar difficulties in producing pandiagonal squares when $r$ is odd and divisible by 3, and have confessed its impossibility when $r$ is even and not divisible by 4. The two methods now brought forward deal with all classes of cases in which general methods for producing pandiagonal squares are supposed to be possible, and introduce for the first time the summation of symmetrical groups as a main object together with the pandiagonal property. In addition to this element which characterizes both methods, that method which relates to odd squares introduces the further novel element of "uniform steps," with an easy rule for steps not uniform; and that which relates to even squares combines the further element, due separately to Franklin, which is needed for what is here called the "complete" square, a square produced at once by the simple process of the "figure of eight."

# *Isotropic Elastic Solids of nearly Spherical Form.*

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## PART II.—VIBRATIONS.

*Table of Contents.*



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### Section I.

§1. The present paper is complementary to a previous one* which dealt with the equilibrium of bodies of nearly spherical form. The method is practically the same, but the differences in detail are considerable. The treatment of the vibration problem is tantamount to assuming that answering to a natural type of vibration in a perfect sphere there is a very similar type of nearly equal frequency in a nearly spherical body. The principal object is to find what may be regarded as the change in pitch due to a small change in the shape of the surface; the result shows what effect an absence of perfect sphericity has on the frequency of vibrations. The present memoir thus does for irregularities in the

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* American Journal of Mathematics, vol. 16, p. 299.

shape of the surface what a previous memoir,* "On Some Compound Vibrating Systems," did for irregularities in the structure of the material.

§2. The first thing required is the complete form of the displacements and stresses for the vibrations natural to a sphere. These I shall simply quote from a previous paper,† slightly altering its notation.

Let $r$, $\theta$, $\phi$ denote the usual polar coordinates, $\theta$ answering to the colatitude, and $\phi$ to the longitude. Let $\Delta$ denote the dilatation, and $u$, $v$, $w$ the displacements at the point $(r, \theta, \phi)$ in the directions of the fundamental elements $dr$, $r\,d\theta$, $r \sin\theta d\phi$. Also for shortness let

$$\alpha^2 = \rho/(m+n), \quad \beta^2 = \rho/n, \tag{1}$$

where $\rho$ is the density and $m$, $n$ the elastic constants in the notation of Thomson and Tait's Natural Philosophy.

Then as types of the dilatation and displacements we have, taking $k/2\pi$ as the frequency of vibration,

$$\Delta = \cos kt\, r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r)\, Y_i \mathbf{Y}_i, \tag{2}$$

$$u = \cos kt \left[\frac{1}{k^2\alpha^2}\left\{\tfrac{1}{2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}} \frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r)\right\} Y_i \mathbf{Y}_i \right.$$
$$\left. + r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\beta r)\, Z_i \mathbf{Z}_i\right], \tag{3}$$

$$v = \cos kt \frac{d}{d\theta}\left[-\frac{1}{k^2\alpha^2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r)\, Y_i \mathbf{Y}_i + \frac{1}{i(i+1)} r^{-1} \frac{d}{dr} r^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, Z_i \mathbf{Z}_i\right]$$
$$+ \cos kt \frac{1}{\sin\theta} \frac{d}{d\phi}\left\{r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, W_i \mathbf{W}_i\right\}, \tag{4}$$

$$w = \cos kt \frac{1}{\sin\theta} \frac{d}{d\phi}\left[-\frac{1}{k^2\alpha^2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r)\, Y_i \mathbf{Y}_i \right.$$
$$\left. + \frac{1}{i(i+1)} r^{-1} \frac{d}{dr} r^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, Z_i \mathbf{Z}_i\right] - \cos kt \frac{d}{d\theta}\left\{r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, W_i \mathbf{W}_i\right\}. \tag{5}$$

Here $J_{i+\frac{1}{2}}(x)$ denotes that solution of the Bessel equation

$$\frac{d^2 z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + \left\{1 - \frac{(i+\frac{1}{2})^2}{x^2}\right\} z = 0$$

which is finite, zero, at the origin $x = 0$.

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*Cambridge Philosophical Society's Transactions, vol. 15, p. 139.

†Cambridge Philosophical Society's Transactions, vol. 14, pp. 310–318.

The heavy letters, $\mathbf{Y}_i$, etc., denote surface harmonics of degree $i$, which may be all the same or all different, while the letters $Y_i$, etc., represent their arbitrary constant multipliers. The $W$ letters answer to the "pure transverse"* vibrations in a perfect sphere; they do not occur at all in $\Delta$. The $Z$ letters and the $Y$ letters with suffixes other than 0, are connected with the type "mixed radial and transverse"* vibrations. The "pure radial"* vibration may be regarded as answering to $Y_0$, with $\mathbf{Y}_0 = 1$.

§3. In exhibiting the values of the stresses, it is convenient to use the following abbreviations:

$$\frac{d}{dx} J(x) = J'(x),$$

$$Y_i \mathbf{Y}_i = [Y_i], \text{ etc.,} \qquad (6)$$

$$\left.\begin{aligned}
&r^{-\frac{1}{2}} \Big[ \frac{(k\beta r)^2 - 2(i-1)(i+2)}{(k\alpha r)^2} J_{i+\frac{1}{2}}(k\alpha r) \\
&\qquad + \frac{2}{k\alpha r} \Big\{ 2J'_{i+\frac{1}{2}}(k\alpha r) - \frac{3}{k\alpha r} J_{i+\frac{1}{2}}(k\alpha r) \Big\} \Big] = {}_rA_i, \\
&r^{-\frac{3}{2}} k\beta r \Big\{ 2J'_{i+\frac{1}{2}}(k\beta r) - \frac{3}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) \Big\} = {}_rB_i, \\
&- r^{-\frac{1}{2}} \frac{1}{k\alpha r} \Big\{ 2J'_{i+\frac{1}{2}}(k\alpha r) - \frac{3}{k\alpha r} J_{i+\frac{1}{2}}(k\alpha r) \Big\} = {}_rC_i, \\
&- r^{-\frac{3}{2}} \frac{k\beta r}{i(i+1)} \Big[ 2J'_{i+\frac{1}{2}}(k\beta r) - \frac{3}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) \\
&\qquad + \Big\{ (k\beta r)^2 - 2(i-1)(1+2) \Big\} \frac{J_{i+\frac{1}{2}}(k\beta r)}{k\beta r} \Big] = {}_rD_i, \\
&r^{-\frac{1}{2}} \Big[ \Big\{ \frac{(k\beta r)^2 - 2}{(k\alpha r)^2} - 2 \Big\} J_{i+\frac{1}{2}}(k\alpha r) \\
&\qquad - \frac{1}{k\alpha r} \Big\{ 2J'_{i+\frac{1}{2}}(k\alpha r) - \frac{3}{k\alpha r} J_{i+\frac{1}{2}}(k\alpha r) \Big\} \Big] = {}_rE_i, \\
&2r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\beta r) = {}_rF_i, \\
&- \frac{2}{(k\alpha r)^2} r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r) = {}_rG_i, \\
&\frac{1}{i(i+1)} r^{-\frac{3}{2}} k\beta r \Big[ 2J'_{i+\frac{1}{2}}(k\beta r) - \frac{3}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) + \frac{4}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) \Big] = {}_rH_i.
\end{aligned}\right\} \quad (7)$$

Using these abbreviations, we have for the typical terms in the stresses employing the expressive notation of Todhunter and Pearson's "History of Elasticity . . . ."

* See Camb. Phil. Soc. Trans., vol. 15, p. 342.

$$\widehat{rr}/n \cos kt = {}_rA_i\,[Y_i] + {}_rB_i\,[Z_i], \tag{8}$$

$$\widehat{\theta\theta}/n \cos kt = {}_rE_i\,[Y_i] + {}_rF_i\,[Z_i] + \frac{d^2}{d\theta^2}\{{}_rG_i\,[Y_i] + {}_rH_i\,[Z_i]\} + \frac{d^2}{d\theta d\phi}\operatorname{cosec}\theta r\,{}_rF_i\,[W_i], \tag{9}$$

$$\widehat{\phi\phi}/n \cos kt = {}_rE_i\,[Y_i] + {}_rF_i\,[Z_i] + \left(\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2} + \cot\theta\frac{d}{d\theta}\right)\{{}_rG_i\,[Y_i] + {}_rH_i\,[Z_i]\} - \frac{d^2}{d\theta d\phi}\operatorname{cosec}\theta r\,{}_rF_i\,[W_i], \tag{10}$$

$$\widehat{r\theta}/n \cos kt = \frac{d}{d\theta}\{{}_rC_i\,[Y_i] + {}_rD_i\,[Z_i]\} + \frac{1}{\sin\theta}\frac{d}{d\phi}\tfrac{1}{2}r\,{}_rB_i\,[W_i], \tag{11}$$

$$\widehat{r\phi}/n \cos kt = \frac{1}{\sin\theta}\frac{d}{d\phi}\{{}_rC_i\,[Y_i] + {}_rD_i\,[Z_i]\} - \frac{d}{d\theta}\tfrac{1}{2}r\,{}_rB_i\,[W_i], \tag{12}$$

$$\widehat{\theta\phi}/n \cos kt = \frac{d^2}{d\theta d\phi}\operatorname{cosec}\theta\{{}_rG_i\,[Y_i] + {}_rH_i\,[Z_i]\} - \left(\frac{d^2}{d\theta^2} + \tfrac{1}{2}i(i+1)\right)r\,{}_rF_i\,[W_i]. \tag{13}$$

§4. The surface will in every case be supposed to approach closely to that of a sphere of radius $a$, and we shall have frequent occasion to employ the values of the stresses when $r = a$. We shall thus for brevity write

$A_i$, etc., for ${}_aA_i$, etc.,
$x$ for $k\alpha a$,
$y$ for $k\beta a$,
$\dfrac{dA_i}{da}$, etc., for the value of $\dfrac{d}{dr}\,{}_rA_i$, etc., with $r$ put $= a$ after differentiation.

The following relations will be found useful:

$$E_i = C_i + (y^2 - 2x^2 - 2)(A_i + 2C_i)/\{y^2 - 2(i-1)(i+2)\}, \tag{14}$$

$$F_i = -2\{B_i + i(i+1)D_i\}/\{y^2 - 2(i-1)(i+2)\}, \tag{15}$$

$$G_i = -2(A_i + 2C_i)/\{y^2 - 2(i-1)(i+2)\}, \tag{16}$$

$$H_i = \left\{\frac{y^2 - 2i(i+1)}{i(i+1)}B_i - 4D_i\right\}/\{y^2 - 2(i-1)(i+2)\}, \tag{17}$$

$$A_iF_i - B_iE_i = \{y^2 - 2(i-1)(i+2)\}^{-1}[-2i(i+1)(A_iD_i - B_iC_i) - B_i\{(y^2 - 2x^2)A_i + (3y^2 - 4x^2)C_i\}, \tag{18}$$

$$A_iH_i - B_iG_i = \{y^2 - 2(i-1)(i+2)\}^{-1}\left[-4(A_iD_i - B_iC_i) + \frac{y^2}{i(i+1)}A_iB_i\right], \quad (19)$$

$$a\frac{dA_i}{da} = \frac{4(y^2 - x^2)(A_i + 2C_i)}{y^2 - 2(i-1)(i+2)} - 3A_i - \tfrac{1}{2}C_i\{y^2 - 2i(i+1)\}, \quad (20)$$

$$a\frac{dB_i}{da} = -4B_i + 2\frac{y^2 - (i-1)(i+2)}{y^2 - 2(i-1)(i+2)}\{B_i + i(i+1)D_i\}, \quad (21)$$

$$a\frac{dC_i}{da} = -4C_i + 2\frac{x^2 - (i-1)(i+2)}{y^2 - 2(i-1)(i+2)}(A_i + 2C_i), \quad (22)$$

$$a\frac{dD_i}{da} = -2D_i - \frac{y^2 - 2i(i+1)}{2i(i+1)}B_i + \frac{\frac{1}{i(i+1)}B_i + D_i}{y^2 - 2(i-1)(i+2)}y^2, \quad (23)$$

$$a\frac{dF_i}{da} = B_i - F_i, \quad (24)$$

$$A_ia\frac{dB_i}{da} - B_ia\frac{dA_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}[2i(i+1)\{y^2 - (i-1)(i+2)\}$$
$$\times(A_iD_i - B_iC_i) - (3y^2 - 4x^2)A_iB_i + \tfrac{1}{2}\{y^4 - 4(3y^2 - 4x^2)\}B_iC_i], \quad (25)$$

$$D_ia\frac{dC_i}{da} - C_ia\frac{dD_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}\Big[2\{x^2 - (i-1)(i+2)\}$$
$$\times(A_iD_i - B_iC_i) - (3y^2 - 4x^2)C_i(\tfrac{1}{2}B_i + D_i) + y^2\left\{\frac{y^2 - (i-1)(i+2)}{2i(i+1)}\right\}B_iC_i\Big], \quad (26)$$

$$D_ia\frac{dA_i}{da} - C_ia\frac{dB_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}[-2\{y^2 - 3(i-1)(i+2)\}$$
$$\times(A_iD_i - B_iC_i) + (3y^2 - 4x^2)D_i(A_i + 2C_i) - \tfrac{1}{2}y^4C_iD_i], \quad (27)$$

$$A_ia\frac{dD_i}{da} - B_ia\frac{dC_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}\Big[-\{y^2 - 4(i-1)(i+2)\}(A_iD_i - B_iC_i)$$
$$+(3y^2 - 4x^2)B_iC_i - \frac{1}{2i(i+1)}\{y^4 - 4(i-1)(i+2)(y^2 - x^2)$$
$$-2(3y^2 - 4x^2)\}A_iB_i\Big], \quad (28)$$

$$A_ia\frac{dD_i}{da} - B_ia\frac{dC_i}{da} + A_iF_i - B_iE_i + (i^2 + i - 3)(A_iH_i - B_iG_i)$$
$$= -\{y^2 - 2(i-1)(i+2)\}^{-1}\Big[\{y^2 + 2(i-1)(i+2)\}(A_iD_i - B_iC_i)$$
$$+ \frac{1}{2i(i+1)}y^2\{y^2 - 4(i-1)(i+2)\}A_iB_i\Big]. \quad (29)$$

The form in which several of these results are put forward is intended to facilitate their calculation when

$$A_iD_i - B_iC_i = 0.$$

§5. In any specified material $x/y$ is to be regarded as a known numerical quantity. Perhaps the simplest material to deal with is that which, though of ordinary rigidity, is nearly incompressible; for it $n$ is finite, but $n/m$, and so $x/y$, vanishingly small. For a vibration of finite frequency in such a material $y$ is finite, but $x$ very small, and in any given expression we need retain only the lowest power of $x$. If we do so we much simplify many of the results. Thus, for instance, putting for shortness

$$a^{-\frac{1}{2}}x^{-\frac{1}{2}}J_{i+\frac{1}{2}}(x) = X,$$

we may take

$$J'_{i+\frac{1}{2}}(x) = \tfrac{1}{2}(2i+1)\,a^{\frac{1}{2}}xX,$$

$$A_i = \{y^2 - 2i(i-1)\}\,X, \tag{30}$$

$$C_i = -2(i-1)\,X, \tag{31}$$

$$A_iD_i - B_iC_i = [\{y^2 - 2i(i-1)\}\,D_i + 2(i-1)\,B_i]\,X, \tag{32}$$

$$A_iF_i - B_iE_i = -\{y^2 - 2(i-1)(i+2)\}^{-1}[2i(i+1)(A_iD_i - B_iC_i) + y^2\{y^2 - 2(i-1)(i+3)\}\,B_iX], \tag{33}$$

$$A_iH_i - B_iG_i = -\{y^2 - 2(i-1)(i+2)\}^{-1}\Big[4(A_iD_i - B_iC_i) - \frac{1}{i(i+1)}\,y^2\{y^2 - 2i(i-1)\}\,B_iX\Big], \tag{34}$$

$$a\frac{dA_i}{da} = i\{y^2 - 2(i-1)(i-2)\}\,X, \tag{35}$$

$$a\frac{dC_i}{da} = -2(i-1)(i-2)\,X. \tag{36}$$

§6. In the case of free vibrations, one of the constants appearing in the expressions for the strains, stresses, etc., may be regarded as arbitrary, being determined by the amplitude of the vibration. The ratios borne to it by the other constants are to be determined from the three surface conditions:

$$\left.\begin{aligned} \lambda\,\widehat{rr} + \mu\,\widehat{r\theta} + \nu\,\widehat{r\phi} &= 0,\\ \lambda\,\widehat{r\theta} + \mu\,\widehat{\theta\theta} + \nu\,\widehat{\theta\phi} &= 0,\\ \lambda\,\widehat{r\phi} + \mu\,\widehat{\theta\phi} + \nu\,\widehat{\phi\phi} &= 0,\end{aligned}\right\} \tag{37}$$

where $\lambda$, $\mu$, $\nu$ are the direction cosines of the outward-drawn normal relative to the fundamental directions $dr$, $rd\theta$, $r\sin\theta d\phi$.

If the equation to the surface be

$$r = a(1 + \epsilon\sigma), \tag{38}$$

where $\sigma$ is a surface harmonic, $\epsilon$ a small positive or negative numerical quantity, we have as a first approximation

$$\lambda = 1, \quad \mu = -\epsilon \frac{d\sigma}{d\theta}, \quad \nu = -\epsilon \frac{1}{\sin\theta} \frac{d\sigma}{d\phi}, \tag{39}$$

while to a second approximation

$$\left.\begin{aligned} \lambda &= 1 - \tfrac{1}{2}\epsilon^2 \left\{\left(\frac{d\sigma}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma}{d\phi}\right)^2\right\}, \\ \mu &= -\epsilon(1-\epsilon\sigma)\frac{d\sigma}{d\theta}, \\ \nu &= -\epsilon(1-\epsilon\sigma)\frac{1}{\sin\theta}\frac{d\sigma}{d\phi}. \end{aligned}\right\} \tag{40}$$

## SECTION II.

§7. The first application of the method is to the case of approximately pure radial vibrations in the material contained by the surface

$$r = a(1 + \epsilon_i \sigma_i), \tag{1}$$

$\sigma_i$ being a surface harmonic of degree $i$, and $\epsilon_i$ a small numerical quantity.

If the surface were truly spherical, the displacement

$$u = \cos kt\, Y_0 \frac{r^{-\frac{1}{2}}}{ka}\left\{\tfrac{1}{2}\frac{J_{\frac{1}{2}}(kar)}{kar} - J'_{\frac{1}{2}}(kar)\right\}, \tag{2}$$

where $Y_0$ is an arbitrary constant, would alone exist. The frequency $k/2\pi$ of the fundamental and higher vibrations would be given by

$$(\widehat{rr})_{r=a} = 0,$$

or

$$_aA_0 = 0. \tag{3}$$

Let us now assume that when (1) is the equation to the surface there is a type of vibration in which (2) is the *principal* term, there being other necessary *subsidiary* terms of the order $\epsilon_i$ of small quantities.

Supposing we go as far as terms in $\epsilon_i^2$, then the contribution of (2) to the first of the surface conditions (37), Section I, is, omitting the $\cos kt$,

$$\left[1 + \epsilon_i\sigma_i a\frac{d}{da} + \tfrac{1}{2}(\epsilon_i\sigma_i)^2 a^2 \frac{d^2}{da^2} - \tfrac{1}{2}\epsilon_i^2\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\right] A_0 Y_0.$$

In this $Y_0$ is treated as a constant, $A_0$ as a function of $a$, the abbreviation $A_i$ for $_aA_i$, etc., being used as in Section I.

Similarly, omitting likewise $\cos kt$, the contributions of (2) to the second and third surface conditions are respectively

$$-\epsilon_i \frac{d\sigma_i}{d\theta}\left(1 - \epsilon_i\sigma_i + \epsilon_i\sigma_i a \frac{d}{da}\right) E_0 Y_0, \text{ and } -\epsilon_i \frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\left(1 - \epsilon_i\sigma_i + \epsilon_i\sigma_i a \frac{d}{da}\right) E_0 Y_0.$$

We can get rid of all the terms of order $\epsilon_i$ in the surface conditions by introducing as *subsidiary* terms those typical terms in (3), (4), (5), Section I, whose coefficients are $Y_i$ and $Z_i$, putting

$$\mathbf{Y}_i = \mathbf{Z}_i = \sigma_i.$$

To get rid of terms of order $\epsilon_i^2$ other harmonic terms would be required, but as the knowledge of these is not actually required for our present purpose, we need only mention their existence.

The first surface condition may now be written

$$\left[1 + \epsilon_i\sigma_i a \frac{d}{da} + \tfrac{1}{2}(\epsilon_i\sigma_i)^2 a^2 \frac{d^2}{da^2} - \tfrac{1}{2}\epsilon_i^2 \left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\right] A_0 Y_0$$
$$+ \left(1 + \epsilon_i\sigma_i a \frac{d}{da}\right)(A_i Y_i + B_i Z_i)\sigma_i - \epsilon_i (C_i Y_i + D_i Z_i)\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}$$
$$+ \text{other harmonic terms of order } \epsilon_i^3 = 0. \qquad (4)$$

For the second surface condition we have

$$-\epsilon_i \frac{d\sigma_i}{d\theta}\left(1 - \epsilon_i\sigma_i + \epsilon_i\sigma_i a \frac{d}{da}\right) E_0 Y_0 + \left(1 + \epsilon_i\sigma_i a \frac{d}{da}\right)\frac{d\sigma_i}{d\theta}(C_i Y_i + D_i Z_i)$$
$$-\epsilon_i \frac{d\sigma_i}{d\theta}\sigma_i (E_i Y_i + F_i Z_i) - \tfrac{1}{2}(G_i Y_i + H_i Z_i)\frac{d}{d\theta}\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}$$
$$+ \text{other harmonics of order } \epsilon_i^3 = 0. \qquad (5)$$

The third surface condition may be derived from the second by writing $\frac{1}{\sin\theta}\frac{d}{d\phi}$ for $\frac{d}{d\theta}$.

In all cases $Y_i$ and $Z_i$ are to be treated as constants during differentiations.

§8. The coefficient of $\sigma_i$ in (4) must vanish, thus we have as a first approximation

$$A_i Y_i + B_i Z_i = -Y_0 \epsilon_i a \frac{dA_0}{da}. \qquad (6)$$

Again the coefficient of $\frac{d\sigma_i}{d\theta}$ in (5)—which is also the coefficient of $\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}$ in the third surface condition—must vanish; thus to a first approximation

$$C_i Y_i + D_i Z_i = Y_0 \epsilon_i E_0. \tag{7}$$

From (6) and (7) we get as first approximations

$$\left.\begin{aligned} Y_i &= -\epsilon_i Y_0\left(D_i a\frac{dA_0}{da} + B_i E_0\right) \div (A_i D_i - B_i C_i), \\ Z_i &= \quad \epsilon_i Y_0\left(C_i a\frac{dA_0}{da} + A_i E_0\right) \div (A_i D_i - B_i C_i). \end{aligned}\right\} \tag{8}$$

Returning to (4) we obtain the frequency equation by equating to zero the terms independent of surface harmonics. Since in the constant terms the coefficients of $Y_i$ and $Z_i$ are of order $\epsilon_i$, it suffices to substitute for these the first approximations (8); before this substitution is effected, the differentiations indicated in (4) have to be carried out.

Since the first approximation to the frequency equation is

$$A_0 = 0,$$

it is obvious that the terms arising from

$$-\tfrac{1}{2}\epsilon_i^2\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\} A_0 Y_0$$

may be neglected.

Thus when powers of $\epsilon_i$ above the second are neglected, we get for the frequency equation, dividing out by $Y_0$,

$$A_0 + \epsilon_i^2\left[\left\{\tfrac{1}{2}a^2\frac{d^2A_0}{da^2} + \frac{a\frac{dB_i}{da}\left(C_i a\frac{dA_0}{da} + A_i E_0\right) - a\frac{dA_i}{da}\left(D_i a\frac{dA_0}{da} + B_i E_0\right)}{A_i D_i - B_i C_i}\right\}\right.$$

$$\left.\times\left\{\text{const. term in } \sigma_i^2\right\} - E_0\left\{\text{const. term in}\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\right] = 0. \tag{9}$$

Supposing the effect of the departure of the surface from the truly spherical form has been to increase the frequency by the amount $\delta k/2\pi$, then $k - \delta k$ must be a solution of the equation

$$A_0 = 0.$$

Consequently, neglecting $\delta k^2$, we must have

$$A_0 - \frac{\delta k}{k}\, k\frac{dA_0}{dk} = 0.$$

But $a^i A_0$ being a function of $ka$, and $A_0$ being to a first approximation zero, we may in the coefficient of $\delta k/k$ put

$$k\frac{dA_0}{dk} = a\frac{dA_0}{da},$$

and thus get

$$A_0 - \frac{\delta k}{k}\, a\frac{dA_0}{da} = 0. \tag{10}$$

The two frequency equations (9) and (10) must be identical, thus

$$\frac{\delta k}{k} = \frac{-\epsilon_i^2}{a\frac{dA_0}{da}}\left[\left\{\tfrac{1}{2}a^2\frac{d^2A_0}{da^2} + \frac{a\frac{dB_i}{da}\left(C_i a\frac{dA_0}{da} + A_i E_0\right) - a\frac{dA_i}{da}\left(D_i a\frac{dA_0}{da} + B_i E_0\right)}{A_i D_i - B_i C_i}\right\}\right.$$

$$\left.\times\left\{\text{const. term in } \sigma_i^2\right\} - E_0\left\{\text{const. term in } \left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\right] \tag{11}$$

§9. It is a simple matter to find the constant terms in $\sigma_i^2$ and

$$\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2 \text{ expanded in surface harmonics.}$$

It is in fact obvious that, putting $\cos\theta = \mu$,

$$\int_{-1}^{+1}\int_0^{2\pi} \sigma_i^2\, d\mu\, d\phi = \int_{-1}^{+1}\int_0^{2\pi} (\text{constant term in } \sigma_i^2)\, d\mu\, d\phi\,;$$

$$\therefore \text{ constant term in } \sigma_i^2 = \frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} \sigma_i^2\, d\mu\, d\phi. \tag{12}$$

The value of this integral is known for the ordinary forms of Laplace Functions. For instance,* if

$$\sigma_i = P_{i,\kappa}(\mu)\cos\kappa\phi = \frac{\lfloor i-\kappa}{1.3\ldots.(2i-1)}(1-\mu^2)^{\frac{\kappa}{2}}\frac{d^\kappa}{d\mu^\kappa}P_i(\mu)\cos\kappa\phi,$$

* See Ferrers' "Spherical Harmonics," pp. 85, 86, or Todhunter's "Functions of Laplace ....," p. 158.

where $P_i$ denotes $i^{\text{th}}$ zonal harmonic and $x$ is integral,

$$\int_{-1}^{+1}\int_0^{2\pi}\sigma_i^2 d\mu\, d\phi = 2\pi \lfloor i-x \lfloor i+x \div [(2i+1)\{1.3\ldots.(2i-1)\}^2].$$

Similarly constant term in $\left(\frac{d\sigma_i}{d\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2$

$$= \frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi}\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\} d\mu\, d\phi$$

$$= \frac{i(i+1)}{4\pi}\int_{-1}^{+1}\int_0^{2\pi}\sigma_i^2 d\mu\, d\phi$$

(see Cambridge Philosophical Society's Transactions, vol. 16, p. 33).

Thus constant term in $\left(\frac{d\sigma_i}{d\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2$

$$= i(i+1)(\text{constant term in } \sigma_i^2). \tag{13}$$

Employing (12) and (13), we throw (11) into the form

$$\frac{\delta k}{k} = -\frac{\epsilon_i^2}{a\frac{dA_0}{da}}\frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi}\sigma_i^2\, d\mu\, d\phi \times\left[\tfrac{1}{2}a^2\frac{d^2A_0}{da^2} - i(i+1)E_0 + \frac{a\frac{dA_0}{da}\left(C_i a\frac{dB_i}{da} - D_i a\frac{dA_i}{da}\right) + E_0\left(A_i a\frac{dB_i}{da} - B_i a\frac{dA_i}{da}\right)}{A_iD_i - B_iC_i}\right]. \tag{14}$$

§10. For a given value of $i$, the change of pitch is the same in all cases in which $\int_{-1}^{+1}\int_0^{2\pi}\sigma_i^2 d\mu\, d\phi$ has a constant value; its sign is independent of that of $\epsilon_i$.

We may simplify (14) by means of the relation

$$\frac{\epsilon_i^2}{4\pi}\int_{-1}^{+1}\int_0^{2\pi}\sigma_i^2 d\mu\, d\phi = \frac{2(\delta S_i/S)}{i^2+i+2}, \tag{15}$$

where $S$ is the area of the surface of a sphere of radius $a$, $S+\delta S_i$ that of the surface (1).

We may also employ the first approximation frequency equation (3) inside the square bracket; doing so, we may put

$$\left.\begin{aligned} a^2\frac{d^2A_0}{da^2}\Big/ a\frac{dA_0}{da} &= -2\frac{(k\alpha a)^2(m+n)^2-8n(3m-n)}{(k\alpha a)^2(m+n)^2-4n(3m-n)} = -2\frac{y^4-8(3y^2-4x^2)}{y^4-4(3y^2-4x^2)}, \\ E_0\Big/ a\frac{dA_0}{da} &= -\frac{2n(3m-n)}{(k\alpha a)^2(m+n)^2-4n(3m-n)} = \frac{-2(3y^2-4x^2)}{y^4-4(3y^2-4x^2)}, \end{aligned}\right\} \quad (16)$$

where $x$ and $y$ have their previous significations.

We may thus replace (14) by

$$\begin{aligned} \frac{\delta k}{k} = \frac{2(\delta S_i/S)}{i^2+i+2}\{y^4-4(3y^2-4x^2)\}^{-1}\Big[y^4 &- 2(i^2+i+4)(3y^2-4x^2) \\ &+ \{y^4-4(3y^2-4x^2)\}\Big(D_i a\frac{dA_i}{da}-C_i a\frac{dB_i}{da}\Big)\Big/(A_iD_i-B_iC_i) \\ &+ 2(3y^2-4x^2)\Big(A_i a\frac{dB_i}{da}-B_i a\frac{dA_i}{da}\Big)\Big/(A_iD_i-B_iC_i)\Big]. \end{aligned} \quad (17)$$

The employment of the results (25) and (27) of Section I renders it unnecessary to actually perform the differentiations occurring on the right of (17). The values to be assigned to $x$ and $y$ in (17) being determined by (3), are to be regarded as known quantities. Professor Lamb* has actually determined the values of $x/\pi$ for the six notes of lowest pitch for a series of values of Poisson's ratio.

## SECTION III.

§11. In a perfect sphere the vibrations next in simplicity to the pure radial are the pure transverse. We thus proceed to consider some vibrations approximately of this type in nearly spherical solids.

The first case we shall consider is when the surface is the nearly spherical spheroid

$$r = a(1+\epsilon_2 P_2), \quad (1)$$

and the principal term in the displacement is

$$w = -W_i \cos kt \frac{dP_i}{d\theta} r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r), \quad (2)$$

where $W_i$ is a constant, and the rest of the notation is as before.

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*Proceedings London Mathematical Society, vol. 18, p. 202; see also Trans. Camb. Phil. Soc., vol. 15, p. 157, or Love's "Treatise on Elasticity," vol. 1, p. 318.

The stresses $\widehat{r\phi}$, $\widehat{\theta\phi}$ answering to (2) would be given over a spherical surface of radius $a$ by

$$\left.\begin{aligned}\widehat{r\phi}/n\cos kt &= -\tfrac{1}{2}W_i a B_i\frac{dP_i}{d\theta},\\ \widehat{\theta\phi}/n\cos kt &= -\tfrac{1}{2}W_i a F_i\left\{i(i+1)P_i+2\frac{d^2P_i}{d\theta^2}\right\},\end{aligned}\right\} \tag{3}$$

where $B_i$ stands for ${}_aB_i$, etc.

In the present instance the change in the frequency has a term of the order $\varepsilon_2$, so we may neglect terms of order $\varepsilon_2^2$ and take for the surface condition

$$\widehat{r\phi}-\varepsilon_2\frac{dP_2}{d\theta}\widehat{\theta\phi}=0. \tag{4}$$

Having regard to (4) and to the relations

$$P_2\frac{dP_i}{d\theta}=\frac{3i(i+1)}{2(2i-1)(2i+1)}\frac{dP_{i-2}}{d\theta}+\frac{i^2+i-3}{(2i-1)(2i+3)}\frac{dP_i}{d\theta}+\frac{3i(i+1)}{2(2i+1)(2i+3)}\frac{dP_{i+2}}{d\theta}, \tag{5}$$

$$\frac{dP_2}{d\theta}\left\{i(i+1)P_i+2\frac{d^2P_i}{d\theta^2}\right\}=\frac{3i(i+1)(i+2)}{(2i-1)(2i+1)}\frac{dP_{i-2}}{d\theta}+\frac{9(i-1)(i+2)}{(2i-1)(2i+3)}\frac{dP_i}{d\theta}-\frac{3(i-1)i(i+1)}{(2i+1)(2i+3)}\frac{dP_{i+2}}{d\theta}, \tag{6}$$

we see that the addition to (2) of subsidiary terms of the same type in $\frac{dP_{i-2}}{d\theta}$ and $\frac{dP_{i+2}}{d\theta}$ is sufficient to solve the problem. We thus assume in place of (2)

$$w=-\cos kt\frac{d}{d\theta}\Big[W_{i-2}P_{i-2}r^{-\frac{1}{2}}J_{i-\frac{3}{2}}(k\beta r)+W_iP_ir^{-\frac{1}{2}}J_{i+\frac{1}{2}}(k\beta r)+W_{i+2}P_{i+2}r^{-\frac{1}{2}}J_{i+\frac{5}{2}}(k\beta r)\Big], \tag{7}$$

where we know $W_{i-2}$, $W_{i+2}$ to be of the order $\varepsilon_2$ of small quantities.

Referring to (3), (5), (6) and (7), we see that the surface condition leads to

$$\begin{aligned}W_i\Big[aB_i\frac{dP_i}{d\theta}&+\varepsilon_2 a\frac{d}{da}(aB_i)\frac{d}{d\theta}\left\{\frac{3i(i+1)}{2(2i-1)(2i+1)}P_{i-2}+\frac{i^2+i-3}{(2i-1)(2i+3)}P_i+\frac{3i(i+1)P_{i+2}}{2(2i+1)(2i+3)}\right\}\\ &+\varepsilon_2 aF_i\frac{d}{d\theta}\left\{-\frac{3i(i+1)(i+2)}{(2i-1)(2i+1)}P_{i-2}-\frac{9(i-1)(i+2)}{(2i-1)(2i+3)}P_i+\frac{3(i-1)i(i+1)}{(2i+1)(2i+3)}P_{i+2}\right\}\Big]\\ &+W_{i-2}aB_{i-2}\frac{dP_{i-2}}{d\theta}+W_{i+2}aB_{i+2}\frac{dP_{i+2}}{d\theta}=0.\end{aligned} \tag{8}$$

As this holds all over the surface, we may equate separately to zero the coefficients of the differentials of the three harmonics. Doing so, we find

$$W_{i-2}/W_i = \frac{3i(i+1)\epsilon_2}{(2i-1)(2i+1)}\left\{(i+2)aF_i - \tfrac{1}{2}a\frac{d}{da}(aB_i)\right\} \Big/ aB_{i-2}. \tag{9}$$

$$aB_i + \frac{\epsilon_2}{(2i-1)(2i+3)}\left\{(i^2+i-3)\,a\frac{d}{da}(aB_i) - 9(i-1)(i+2)\,aF_i\right\} = 0, \tag{10}$$

$$W_{i+2}/W_i = -\frac{3i(i+1)\epsilon_2}{(2i+1)(2i+3)}\left\{(i-1)\,aF_i + \tfrac{1}{2}a\frac{d}{da}(aB_i)\right\} \Big/ aB_{i+2}. \tag{11}$$

The value of $W_i$ is supposed to be given by the known amplitude of vibration, so that (9) and (11) determine $W_{i-2}$ and $W_{i+2}$.

§12. For the frequency equation we have (10), which becomes for the perfect sphere

$$B_i = 0. \tag{12}$$

Reasoning similar to that in Section II leads us to the conclusion that if $\delta k/2\pi$ be the increment in the pitch due to the departure from the spherical form, the equation

$$aB_i - \frac{\delta k}{k}\,a\frac{d}{da}(aB_i) = 0$$

must be identical with (10).

Thus we get

$$\frac{\delta k}{k} = -\frac{\epsilon_2}{(2i-1)(2i+3)}\left\{i^2+i-3-9(i-1)(i+2)\frac{aF_i}{a\dfrac{d}{da}(aB_i)}\right\}. \tag{13}$$

In the coefficient of $\epsilon_2$ we may regard $k$ as possessing the values it would have in a perfect sphere, i. e. as being a root of (12); and it is easily shown from (15) and (21), Section I, that when $B_i = 0$,

$$a\frac{dB_i}{da} = -F_i\{y^2-(i-1)(i+2)\}. \tag{14}$$

Employing this result in (13) we get

$$\frac{\delta k}{k} = -\epsilon_2\frac{(i^2+i-3)y^2-(i-3)(i-1)(i+2)(i+4)}{(2i-1)(2i+3)\{y^2-(i-1)(i+2)\}}, \tag{15}$$

where $y \equiv k\beta a$ is the root of

$$2yJ'_{i+\frac{1}{2}}(y) - 3J_{i+\frac{1}{2}}(y) = 0, \tag{16}$$

which answers to the note of the transverse type under consideration.

§13. The functions $J_{i+\frac{1}{2}}(y)$ take conveniently concise forms involving sines and cosines. These will be found for values of $i$ up to 5 on p. 42 of Gray and Mathews' "Treatise on Bessel Functions." Their substitution will be found to lead to comparatively simple frequency equations for small values of $i$. The equations answering to $i = 1$ and $i = 2$ were discussed by Professor Lamb in his valuable paper in the Proc. of the London Math. Soc., vol. 13 (see his equations (38) and (43)), and the numerical values of the smaller roots will be found there and in Love's "Treatise on .... Elasticity," vol. 1, pp. 318–9.

In the case $i = 1$ we need not trouble about the form or roots of (16) for our present purpose, because (15) reduces to the simple form

$$\delta k/k = \epsilon_2/5. \tag{17}$$

The pitch is thus altered in the same proportion in all the notes; it is raised or lowered according as the surface is prolate or oblate.

For $i = 2$, (16) takes the form

$$\tan y = y\,(y^2 - 12)/(5y^2 - 12), \tag{18}$$

and the six lowest notes are approximately given by

$$y/\pi = .7961,\ 2.2715,\ 3.3469,\ 4.3837,\ 5.4059,\ 6.4209.$$

In this case (15) becomes

$$\frac{\delta k}{k} = -\frac{\epsilon_2}{7}\,\frac{y^2+8}{y^2-4}. \tag{19}$$

It follows from the above values of the roots that $\delta k$ is here always opposite in sign to $\epsilon_2$, or the pitch is raised or lowered according as the spheroid is oblate or prolate. For the higher notes, when $i = 2$,

$$\delta k/k = -\epsilon_2/7 \tag{20}$$

becomes an increasingly close approximation.

For all values of $i$ successive higher roots of (16) become approximately successive even or successive odd multiples of $\pi/2$. Thus when $i$ is not large we get from (15) as an approximation for all the higher "harmonics"

$$\delta k/k = -\epsilon_2 (i^2 + i - 3)/\{(2i-1)(2i+3)\}, \tag{21}$$

a result which, of course, includes (20).

On the other hand, when $i$ is very large we get as an approximation in the case of the fundamental note and lower harmonics

$$\delta k/k = -\epsilon_2/4. \tag{22}$$

This last result, and likewise (21) when $i$ exceeds 1, represent a change of pitch opposite in sign to $\epsilon_2$.

## SECTION IV.

§14. As a second example of transverse vibrations, consider the solid contained by the approximately spherical surface

$$r = a(1 + \epsilon_i P_i) \tag{1}$$

vibrating approximately in the *rotatory** type given by

$$w/\cos kt = w_1 \sin\theta\, r^{-\frac{1}{2}} J_{\frac{3}{2}}(k\beta r), \tag{2}$$

where $w_1$ is a constant.

Answering to a displacement

$$w/\cos kt = -W_j \frac{dP_j}{d\theta} r^{-\frac{1}{2}} J_{j+\frac{1}{2}}(k\beta r), \tag{3}$$

where $W_j$ is a constant, we have over $r = a$,

$$\widehat{r\phi}/n \cos kt = -\tfrac{1}{2} W_j a B_j \frac{dP_j}{d\theta}, \tag{4}$$

$$\widehat{\theta\phi}/n \cos kt = \tfrac{1}{2} W_j a F_j \left\{ j(j+1) P_j - 2\mu \frac{dP_j}{d\mu} \right\}, \tag{5}$$

where $\mu \equiv \cos\theta$.

For $j = 1$, $\widehat{\theta\phi}$ vanishes and so the contribution of (2) to the surface condition

$$\widehat{r\phi} - \epsilon_i \frac{dP_i}{d\theta} \widehat{\theta\phi} = 0 \tag{6}$$

is only

$$\left(1 + \epsilon_i P_i a \frac{d}{da}\right) W_1 \sin\theta\, a B_1/2,$$

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*Professor Lamb's term for case $i = 1$.

i. e.*
$$\left\{1+\frac{\varepsilon_i}{2i+1}\frac{d}{d\mu}(P_{i+1}-P_{i-1})\,a\frac{d}{da}\right\}W_1\sin\theta\, aB_1/2.$$

It is thus obvious that when we neglect terms in $\varepsilon_i^2$ it suffices to add to the value of $w/\cos kt$ given in (2) the subsidiary terms

$$-\frac{d}{d\theta}\left[W_{i-1}P_{i-1}r^{-\frac{1}{2}}J_{i-\frac{1}{2}}(k\beta r)+W_{i+1}P_{i+1}r^{-\frac{1}{2}}J_{i+\frac{3}{2}}(k\beta r)\right],$$

where $W_{i-1}$, $W_{i+1}$ are constants.

From principal and subsidiary terms combined we get as the surface condition

$$aB_1+\varepsilon_i a\frac{d}{da}(aB_1)\frac{1}{2i+1}\frac{d}{d\mu}(P_{i+1}-P_{i-1})$$
$$+(W_{i-1}/W_1)aB_{i-1}\frac{dP_{i-1}}{d\mu}+(W_{i+1}/W_1)aB_{i+1}\frac{dP_{i+1}}{d\mu}. \quad (7)$$

As this holds all over the surface, we may equate separately to zero the coefficients of $\frac{dP_{i-1}}{d\mu}$, $\frac{dP_{i+1}}{d\mu}$ and the constant term. We thus find

$$W_{i-1}=\quad\frac{1}{2i+1}W_1\varepsilon_i\frac{\frac{d}{da}(aB_1)}{B_{i-1}}, \quad (8)$$

$$W_{i+1}=-\frac{1}{2i+1}W_1\varepsilon_i\frac{\frac{d}{da}(aB_1)}{B_{i+1}}, \quad (9)$$

and finally for the frequency equation,

$$aB_1=0. \quad (10)$$

Since (10) is identical with the frequency equation in the perfect sphere, $r=a$, the change of frequency in the present case is zero when terms of order $\varepsilon_i^2$ are neglected.

This at first sight appears inconsistent with the result obtained in (17) of Section III, that in the special case of the surface

$$r=a(1+\varepsilon_2 P_2)$$

there is, in the case of the vibrations of approximately rotatory type, a change of pitch given by

$$\delta k/k=\tfrac{1}{5}\varepsilon_2.$$

* See equation (11), p. 46 of Todhunter's "Functions of Laplace . . . ."

The explanation is simply that $i=2$ constitutes an exceptional case. The work of the present section in reality tacitly assumes that $\frac{dP_1}{d\mu}$, $\frac{dP_{i-1}}{d\mu}$ and $\frac{dP_{i+1}}{d\mu}$ are all different, which is not true when $i=2$. The case $i=2$ must thus be excluded from the range of the present section; it clearly is the only case in which the tacit assumption is not true.

§15. When we desire to go as far as terms in $\epsilon_i^2$, we get in place of (7)

$$W_1\left\{aB_1+\tfrac{1}{2}\epsilon_i^2a^2\frac{d^2}{da^2}(aB_1)\times\text{constant term in }P_i^2\right\}$$
$$+\text{constant terms in}\Big[\epsilon_iP_ia\frac{d}{da}\Big\{W_{i-1}aB_{i-1}\frac{dP_{i-1}}{d\mu}+W_{i+1}aB_{i+1}\frac{dP_{i+1}}{d\mu}\Big\}$$
$$+\epsilon_i\frac{dP_i}{d\mu}W_{i-1}aF_{i-1}\Big\{(i-1)\,iP_{i-1}-2\mu\frac{dP_{i-1}}{d\mu}\Big\}$$
$$+\epsilon_i\frac{dP_i}{d\mu}W_{i+1}aF_{i+1}\Big\{(i+1)(i+2)\,P_{i+1}-2\mu\frac{dP_{i+1}}{d\mu}\Big\}\Big]=0. \qquad (11)$$

By the "constant term" in $P_i^2$, for instance, is here meant the value of $A_1$ in the series

$$P_i^2=A_1+A_3\frac{dP_3}{d\mu}+\ldots.+A_{2i+1}\frac{dP_{2i+1}}{d\mu},$$

or

$$P_i^2=(A_1+A_3+\ldots.A_{2i+1})+5P_2(A_3+\ldots.+A_{2i+1})+\ldots.$$
$$=C_0+C_2P_2+\ldots., \text{ say.}$$

To determine $A_1$ we notice that

$$C_0=\tfrac{1}{2}\int_{-1}^{+1}P_i^2\,d\mu=\frac{1}{2i+1}. \qquad (12)$$

Again,

$$C_2\int_{-1}^{+1}P_2^2\,d\mu=\int_{-1}^{+1}P_2P_i^2\,d\mu. \qquad (13)$$

Now it is easy to prove

$$P_2P_i=\frac{3(i-1)\,i}{2(2i-1)(2i+1)}P_{i-2}+\frac{i(i+1)}{(2i-1)(2i+3)}P_i+\frac{3(i+1)(i+2)}{2(2i+1)(2i+3)}P_{i+2};$$

therefore

$$\int_{-1}^{+1}P_2P_i^2d\mu=\frac{i(i+1)}{(2i-1)(2i+3)}\int_{-1}^{+1}P_i^2d\mu$$
$$=2i(i+1)\div\{(2i-1)(2i+1)(2i+3)\}.$$

Hence by (13)

$$C_2 = 5i(i+1) \div \{(2i-1)(2i+1)(2i+3)\}. \quad (14)$$

But by the equivalence of the expressions for $P_i^2$,

$$A_1 = C_0 - \tfrac{1}{3} C_2,$$

$$\therefore\ A_1 = 3(i^2 + i - 1) \div \{(2i-1)(2i+1)(2i+3)\}. \quad (15)$$

In a somewhat similar fashion we find, with like forms of expansion in *differentials* of harmonics,

$$\text{constant term in } P_i \frac{dP_{i-1}}{d\mu} = -3(i-1)i \div \{2(2i-1)(2i+1)\}, \quad (16)$$

$$\text{constant term in } P_i \frac{dP_{i+1}}{d\mu} = 3(i+1)(i+2) \div \{2(2i+1)(2i+3)\}, \quad (17)$$

$$\text{constant term in } \frac{dP_i}{d\mu}\left\{i(i-1)P_{i-1} - 2\mu\frac{dP_{i-1}}{d\mu}\right\} = \frac{3(i-2)(i-1)i(i+1)}{2(2i-1)(2i+1)}, \quad (18)$$

$$\text{constant term in } \frac{dP_i}{d\mu}\left\{(i+1)(i+2)P_{i+1} - 2\mu\frac{dP_{i+1}}{d\mu}\right\}$$

$$= -\frac{3i(i+1)(i+2)(i+3)}{2(2i+1)(2i+3)}. \quad (19)$$

We thus convert (11) into

$$aB_1 + \tfrac{3}{2}\epsilon_i^2 \frac{i^2+i-1}{(2i-1)(2i+1)(2i+3)} a^2 \frac{d^2}{da^2}(aB_1)$$

$$- \epsilon_i (W_{i-1}/W_1) \frac{3(i-1)i}{2(2i-1)(2i+1)} \left\{ a\frac{d}{da}(aB_{i-1}) - (i-2)(i+1)aF_{i-1} \right\}$$

$$\mp \epsilon_i (W_{i+1}/W_1) \frac{3(i+1)(i+2)}{2(2i+1)(2i+3)} \left\{ a\frac{d}{da}(aB_{i+1}) - i(i+3)aF_{i+1} \right\} = 0.$$

Substituting from (8) and (9), we find

$$aB_1 - \frac{3\epsilon_i^2}{2(2i+1)}\Bigg[ -\frac{i^2+i-1}{(2i-1)(2i+3)} a^2 \frac{d^2}{da^2}(aB_1)$$

$$+ \frac{(i-1)i}{(2i-1)(2i+1)} \frac{a\frac{d}{da}(aB_1)}{aB_{i-1}} \left\{ a\frac{d}{da}(aB_{i-1}) - (i-2)(i+1)aF_{i-1} \right\}$$

$$+ \frac{(i+1)(i+2)}{(2i+1)(2i+3)} \frac{a\frac{d}{da}(aB_1)}{aB_{i+1}} \left\{ a\frac{d}{da}(aB_{i+1}) - i(i+3)aF_{i+1} \right\} \Bigg] = 0.$$

§16. Reasoning as before from the identity of this with

$$aB_1 - \frac{\delta k}{k} a \frac{d}{da}(aB_1) = 0,$$

we find for the increment in pitch due to the departure from the truly spherical form

$$\frac{\delta k}{k} = \frac{3}{2} \frac{\epsilon_i^2}{2i+1} \left[ -\frac{i^2+i-1}{(2i-1)(2i+3)} \frac{a^2 \frac{d^2}{da^2}(aB_1)}{a\frac{d}{da}(aB_1)} + \frac{(i-1)i}{(2i-1)(2i+1)} \frac{a\frac{d}{da}(aB_{i-1}) - (i-2)(i+1)aF_{i-1}}{aB_{i-1}} + \frac{(i+1)(i+2)}{(2i+1)(2i+3)} \frac{a\frac{d}{da}(aB_{i+1}) - i(i+3)aF_{i+1}}{aB_{i+1}} \right]. \quad (20)$$

In the coefficient of $\epsilon_i^2$ we substitute for $k$, after differentiation, that root of (10) which gives the first approximation to the frequency of the vibration under consideration. For any such value of $k$ it is easy to prove

$$a^2 \frac{d^2}{da^2}(aB_1) \div a\frac{d}{da}(aB_1) = -2.$$

Also in general from (15), (21), etc., Section I,

$$\frac{a\frac{d}{da}(aB_j) - (j-1)(j+2)aF_j}{aB_j} = -3 - y^2 F_j / B_j.$$

Thus we can replace (20) by the more concise formula

$$\frac{\delta k}{k} = -\frac{3}{2} \frac{\epsilon_i^2}{2i+1} \left[ \frac{4(i^2+i-1)}{(2i-1)(2i+3)} + \frac{y^2}{2i+1} \left\{ \frac{(i-1)iF_{i-1}}{(2i-1)B_{i-1}} + \frac{(i+1)(i+2)F_{i+1}}{(2i+3)B_{i+1}} \right\} \right]. \quad (21)$$

This can be further simplified by the result

$$B_s / F_s = \{yJ'_{s+\frac{1}{2}}(y)\} / J_{s+\frac{1}{2}}(y) - \frac{3}{2}, \quad (22)$$

deducible at once from (7), Section I.

The values to be ascribed to $y$ are the roots of (10), whose approximate values* are given by

$$y/\pi = 1\cdot8346,\ 2\cdot8950,\ 3\cdot9225,\ 4\cdot9385,\ 5\cdot9489,\ 6\cdot9563, \text{ etc.}$$

The algebraical expressions for the Bessels involved for value of $s$ from 1 to 5 are given, as already stated, on p. 42 of Gray and Mathews' Treatise, but in the absence of a complete table of numerical values, the numerical evaluation of $\delta k/k$ from (21) would prove somewhat tedious.

## SECTION V.

§17. The third and most complicated type of vibration in the perfect sphere is the "mixed radial and transverse." As an example in which the principal terms are of this type, we may consider the vibrations in the nearly spherical spheroid

$$r = a(1 + \varepsilon_2 P_2), \tag{1}$$

for which the principal terms in the displacements are those with the coefficients $Y_i$, $Z_i$ in (3), (4) and (5), Section I. We shall take the simplest case when

$$\mathbf{Y}_i = \mathbf{Z}_i = P_i.$$

The requisite subsidiary terms are also of the mixed radial and transverse types, and depend on the harmonics $P_{i-2}$ and $P_{i+2}$.

When terms of order $\varepsilon_2^2$ are neglected, it will be seen from (8) to (13) and (37) of Section I that the first surface equation is

$$\begin{aligned}\left(1 + \varepsilon_2 P_2 a\frac{d}{da}\right)[P_i(A_iY_i + B_iZ_i)] - \varepsilon_2\frac{dP_2}{d\theta}\frac{dP_i}{d\theta}(C_iY_i + D_iZ_i)\\ + P_{i-2}(A_{i-2}Y_{i-2} + B_{i-2}Z_{i-2}) + P_{i+2}(A_{i+2}Y_{i+2} + B_{i+2}Z_{i+2}) = 0.\end{aligned} \tag{2}$$

In like manner the second surface condition is

$$\begin{aligned}\left(1 + \varepsilon_2 P_2 a\frac{d}{da}\right)\frac{dP_i}{d\theta}(C_iY_i + D_iZ_i)\\ - \varepsilon_2\frac{dP_2}{d\theta}\left\{P\,(E_iY_i + F_iZ_i) + \frac{d^2P_i}{d\theta^2}(G_iY_i + H_iZ_i)\right\}\\ + \frac{dP_{i-2}}{d\theta}(C_{i-2}Y_{i-2} + D_{i-2}Z_{i-2}) + \frac{dP_{i+2}}{d\theta}(C_{i+2}Y_{i+2} + D_{i+2}Z_{i+2}) = 0.\end{aligned} \tag{3}$$

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*See Love's Treatise, vol. I, p. 318.

The third surface condition, as there is symmetry round $\theta = 0$, is of course identically satisfied.

In dealing with (2) and (3) we require the values of $P_2 P_i$ and $\frac{dP_2}{d\theta}\frac{dP_i}{d\theta}$ in terms of spherical harmonics, and the values of $P_2\frac{dP_i}{d\theta}$, $P_i\frac{dP_2}{d\theta}$ and $\frac{dP_2}{d\theta}\frac{d^2P_i}{d\theta^2}$ in terms of differential coefficients of spherical harmonics, as given in equations (30) to (34), p. 306, vol. 16 of this journal. Employing these, we deduce from (2), by equating separately to zero the coefficients of $P_{i-2}$, $P_i$ and $P_{i+2}$, the following three equations:

$$A_{i-2}Y_{i-2} + B_{i-2}Z_{i-2} = \frac{3\varepsilon_2 i(i-1)}{(2i-1)(2i+1)}\left\{-\tfrac{1}{2}a\frac{d}{da}(A_iY_i+B_iZ_i) + (i+1)(C_iY_i+D_iZ_i)\right\}, \quad (4)$$

$$A_iY_i + B_iZ_i + \varepsilon_2\frac{i(i+1)}{(2i-1)(2i+3)}\left\{a\frac{d}{da}(A_iY_i+B_iZ_i) - 3(C_iY_i+D_iZ_i)\right\} = 0, \quad (5)$$

$$A_{i+2}Y_{i+2} + B_{i+2}Z_{i+2} = -\frac{3\varepsilon_2(i+1)(i+2)}{(2i+1)(2i+3)}\left\{\tfrac{1}{2}a\frac{d}{da}(A_iY_i+B_iZ_i) + i(C_iY_i+D_iZ_i)\right\}. \quad (6)$$

Similarly treating (3) and equating separately to zero the coefficients of $\frac{dP_{i-2}}{d\theta}$, $\frac{dP_i}{d\theta}$ and $\frac{dP_{i+2}}{d\theta}$, we get

$$C_{i-2}Y_{i-2} + D_{i-2}Z_{i-2} = \frac{3\varepsilon_2 i}{(2i-1)(2i+1)}\Big[-\tfrac{1}{2}(i+1)a\frac{d}{da}(C_iY_i+D_iZ_i) - (E_iY_i+F_iZ_i) + (i+1)^2(G_iY_i+H_iZ_i)\Big], \quad (7)$$

$$C_iY_i + D_iZ_i + \frac{\varepsilon_2}{(2i-1)(2i+3)}\Big[(i^2+i-3)a\frac{d}{da}(C_iY_i+D_iZ_i) - 3(E_iY_i+F_iZ_i) - 3(i^2+i-3)(G_iY_i+H_iZ_i)\Big] = 0, \quad (8)$$

$$C_{i+2}Y_{i+2} + D_{i+2}Z_{i+2} = \frac{3\varepsilon_2(i+1)}{(2i+1)(2i+3)}\Big[-\tfrac{1}{2}ia\frac{d}{da}(C_iY_i+D_iZ_i) + E_iY_i + F_iZ_i - i^2(G_iY_i+H_iZ_i)\Big]. \quad (9)$$

§18. We see from (5) and (8) that in a perfect sphere the frequency equation arises from the elimination of $Y_i$ and $Z_i$ between the two equations

$$A_i Y_i + B_i Z_i = 0, \tag{10}$$

$$C_i Y_i + D_i Z_i = 0, \tag{11}$$

and so is

$$A_i D_i - B_i C_i = 0. \tag{12}$$

This supplies of course the first approximations to the values of $k$ in the present problem.

Use may be made of (10), (11) and (12) to simplify any terms containing $\varepsilon_2$; in particular, $\varepsilon_2 (C_i Y_i + D_i Z_i)$ may be neglected in (4), (5) and (6).

We may obviously determine in terms of $Y_i$ the constants $Y_{i-2}$, $Z_{i-2}$ from (4) and (7), the constants $Y_{i+2}$, $Z_{i+2}$ from (6) and (9).

For the frequency equation we eliminate $Y_i$ and $Z_i$ between (5) and (8), obtaining on reduction

$$A_i D_i - B_i C_i + \frac{\varepsilon_2}{(2i-1)(2i+3)} \Big[ i(i+1)\Big(D_i a \frac{dA_i}{da} - C_i a \frac{dB_i}{da}\Big) + (i^2+i-3)\Big(A_i a \frac{dD_i}{da} - B_i a \frac{dC_i}{da}\Big) - 3(A_i F_i - B_i E_i) - 3(i^2+i-3)(A_i H_i - B_i G_i)\Big] = 0. \tag{13}$$

Reasoning from (12) and (13) as before, we deduce for the increment $\delta k/2\pi$ in pitch, due to the departure of (1) from the truly spherical form, the result

$$\frac{\delta k}{k} = \frac{-3\varepsilon_2}{(2i-1)(2i+3)} \left[ \frac{i(i+1)}{3} - \frac{A_i a \dfrac{dD_i}{da} - B_i a \dfrac{dC_i}{da} + A_i F_i - B_i E_i + (i^2+i-3)(A_i H_i - B_i G_i)}{a \dfrac{d}{da}(A_i D_i - B_i C_i)} \right]. \tag{14}$$

Referring to (27), (28) and (29), Section I, and making use of (12), we find on reduction

$$\frac{\delta k}{k} = \frac{-3\varepsilon_2}{(2i-1)(2i+3)} \left[ \frac{i(i+1)}{3} - \frac{y^2\{y^2 - 4(i-1)(i+2)\} A_i^2}{y^2\{y^2 - (i-1)(i+2)\} A_i^2 + i(i+1) y^4 C_i^2 - i(i+1)(3y^2 - 4x^2)(A_i + 2C_i)^2} \right]. \tag{15}$$

In (15) we are to substitute for $x$ and $y$ the values they possess for the root of (12) under consideration; the value of $m/n$, and so of $x/y$, being supposed of course known.

§19. For most values of $m/n$ the determination of the roots of (12), and of the corresponding values of $A_i/C_i$ to be used in (15), would unquestionably be tedious. The special case of a nearly incompressible material of finite rigidity is however comparatively simple. For it, using (30) and (31) of Section I, we obtain at once from (15)

$$\frac{\delta k}{k}=\frac{-3\epsilon_2}{(2i-1)(2i+3)}\left[\frac{i(i+1)}{3}-\frac{\{y^2-2i(i-1)\}^2\{y^2-4(i-1)(i+2)\}}{\{y^6-2y^4(2i-1)(2i+1)+12y^2i(i-1)(i+1)(2i+1)-8i(i-1)^2(i+2)(2i^2+4i+3)\}}\right]. \quad (16)$$

Here $y$ is a root of

$$\frac{y^4-y^2(2i-1)(2i+1)+2(i-1)i(i+2)(2i+1)}{y^2-2(i-1)i(i+2)}+\frac{2yJ'_{i+\frac{1}{2}}(y)}{J_{i+\frac{1}{2}}(y)}=0, \quad (17)$$

which is the form taken by (12) when $x/y=0$.

For values of $i$ up to 5 the values of $2yJ'_{i+\frac{1}{2}}(y)/J_{i+\frac{1}{2}}(y)$ are obtained at once from the table on p. 42 of Gray and Mathews' Treatise, and the corresponding frequency equations it will be found are by no means very formidable. Thus for $i=1$, (17) takes the form

$$\tan y=y(y^2-6)\div\{3(y^2-2)\}, \quad (18)$$

and for $i=2$,

$$\tan y=-y(5y^4-92y^2+480)/(y^6-25y^4+252y^2-480). \quad (19)$$

These equations were originally given by Professor Lamb in his paper already referred to (Proc. London Math. Soc., vol. 13), and in the same source will be found the approximate numerical values of the lower roots, viz. for (18),

$$y/\pi=1{\cdot}2319,\ 2{\cdot}3692,\ 3{\cdot}4101,\ 4{\cdot}4310,\ 5{\cdot}4439,\ 6{\cdot}4528,$$

and for (19),

$$y/\pi=0{\cdot}8485,\ 1{\cdot}7420,\ 2{\cdot}8257,\ 3{\cdot}8709.$$

For $i=1$ we deduce from (16)

$$\frac{\delta k}{k}=\tfrac{1}{5}\epsilon_2\frac{y^2+12}{y^2-6}, \quad (20)$$

and for $i = 2$,*

$$\frac{\delta k}{k} = -\tfrac{1}{7}\epsilon_2 \frac{y^6 - 36y^4 + 576y^2 - 2176}{y^6 - 30y^4 + 360y^2 - 1216}. \tag{21}$$

Numerical values of $\delta k/k$ are thus easily obtainable in these two cases for the several values of $y$ recorded above.

It is obvious when $i = 1$ that $\delta k/k$ is even for the lowest note of the same sign as $\epsilon_2$. But when $i = 2$, $\delta k/k$ is always opposite in sign to $\epsilon_2$.

In the general case of incompressible material, when $i$ is not very large, we deduce from (16) for very high notes—i. e. for large values of $y$—as an approximation

$$\delta k/k = -\epsilon_2(i^2 + i - 3) \div \{(2i-1)(2i+3)\}. \tag{22}$$

Here when $i$ exceeds 1, the sign of $\delta k/k$ is opposite to that of $\epsilon_2$, or the pitch is raised or lowered according as the spheroid is oblate or prolate.

The identity of (22) with the corresponding result, (21), Section III, for transverse vibrations is noteworthy.

## SECTION VI.

§20. The case of approximately radial and transverse vibrations depending on $P_2$ in a spheroid presents a certain peculiarity inasmuch as $P_{2-2}$ or $P_0$ is a constant. I do not propose, however, to consider this case specially, but pass on to a more general case of an ellipsoid which illustrates equally this peculiarity.

Use will be made of the notation

$$P_{i,s}(\mu) = \frac{\lfloor i - s}{1.3\ldots.(2i-1)}(1-\mu^2)^{\frac{s}{2}}\frac{d^s}{d\mu^s}P_i(\mu), \tag{1}$$

where $\mu \equiv \cos\theta$.
Also for brevity let

$$P_{2,2}\cos 2\phi \equiv \sin^2\theta\cos 2\phi = X_{2,2}. \tag{2}$$

I propose to consider the vibrations which in a perfect sphere would be of

* In strictness the application of (16) when $i = 2$ is not fully warranted at the present stage, see §20; it is however justified by §23, and for convenient comparison with the case $i = 1$ it is introduced here.

the type radial and transverse, involving the harmonics $P_2$ and $X_{2,2}$, taking place in the nearly spherical solid contained by

$$r = a(1 + \epsilon P_2 + \epsilon_2 X_{2,2}). \tag{3}$$

Squares and products of $\epsilon$ and $\epsilon_2$ will be neglected. The lengths $a'$, $b'$, $c'$ of the semi-axes of the ellipsoid (3) are given by

$$\left.\begin{aligned} a'/a &= 1 - \tfrac{1}{2}\epsilon + \epsilon_2, \\ b'/a &= 1 - \tfrac{1}{2}\epsilon - \epsilon_2, \\ c'/a &= 1 + \epsilon. \end{aligned}\right\} \tag{4}$$

The surface conditions take the form

$$\left.\begin{aligned} \widehat{rr} - \frac{d}{d\theta}(\epsilon P_2 + \epsilon_2 X_{2,2})\,\widehat{r\theta} - \frac{1}{\sin\theta}\frac{d}{d\phi}(\epsilon_2 X_{2,2})\,\widehat{r\phi} &= 0, \\ \widehat{r\theta} - \frac{d}{d\theta}(\epsilon P_2 + \epsilon_2 X_{2,2})\,\widehat{\theta\theta} - \frac{1}{\sin\theta}\frac{d}{d\phi}(\epsilon_2 X_{2,2})\,\widehat{\theta\phi} &= 0, \\ \widehat{r\phi} - \frac{d}{d\theta}(\epsilon P_2 + \epsilon_2 X_{2,2})\,\widehat{\theta\phi} - \frac{1}{\sin\theta}\frac{d}{d\phi}(\epsilon_2 X_{2,2})\,\widehat{\phi\phi} &= 0. \end{aligned}\right\} \tag{5}$$

In addition to principal terms representing radial and transverse vibrations depending on $P_2$ and $X_{2,2}$, we require subsidiary terms representing pure radial vibrations, subsidiary terms representing radial and transverse vibrations depending on $P_4$, $P_{4,2}\cos 2\phi$ and $P_{4,4}\cos 4\phi$, and lastly, subsidiary terms representing pure transverse vibrations depending on $P_{3,2}\sin 2\phi$. The complete values of the displacements are as follows:

$$\begin{aligned} u/\cos kt = \frac{r^{-\frac{3}{2}}}{2k^2a^2}\,&[\{J_{\frac{1}{2}}(kar) - 2karJ'_{\frac{1}{2}}(kar)\}\,Y_0 \\ &+ \{J_{\frac{5}{2}}(kar) - 2karJ'_{\frac{5}{2}}(kar)\}(Y_2P_2 + Y_{2,2}X_{2,2}) \\ &+ \{J_{\frac{9}{2}}(kar) - 2karJ'_{\frac{9}{2}}(kar)\}(Y_4P_4 + Y_{4,2}P_{4,2}\cos 2\phi + Y_{4,4}P_{4,4}\cos 4\phi)] \\ &+ r^{-\frac{1}{2}}[J_{\frac{5}{2}}(k\beta r)(Z_2P_2 + Z_{2,2}X_{2,2}) \\ &+ J_{\frac{9}{2}}(k\beta r)(Z_4P_4 + Z_{4,2}P_{4,2}\cos 2\phi + Z_{4,4}P_{4,4}\cos 4\phi)], \end{aligned} \tag{6}$$

$$v/\cos kt = -\frac{r^{-\frac{1}{2}}}{k^2\alpha^2}\frac{d}{d\theta}\left[J_{\frac{5}{2}}(k\alpha r)(Y_2P_2 + Y_{2,2}X_{2,2})\right.$$

$$\left. + J_{\frac{9}{2}}(k\alpha r)(Y_4P_4 + Y_{4,2}P_{4,2}\cos 2\phi + Y_{4,4}P_{4,4}\cos 4\phi)\right]$$

$$+ \frac{1}{r}\frac{d^2}{d\theta dr}r^{\frac{1}{2}}\left\{\tfrac{1}{6}J_{\frac{5}{2}}(k\beta r)(Z_2P_2 + Z_{2,2}X_{2,2})\right.$$

$$\left. + \tfrac{1}{20}J_{\frac{9}{2}}(k\beta r)(Z_4P_4 + Z_{4,2}P_{4,2}\cos 2\phi + Z_{4,4}P_{4,4}\cos 4\phi)\right\}$$

$$+ r^{-\frac{1}{2}}\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{W_{3,2}P_{3,2}\sin 2\phi\, J_{\frac{7}{2}}(k\beta r)\right\}, \qquad (7)$$

$$w/\cos kt = -\frac{r^{-\frac{1}{2}}}{k^2\alpha^2}\frac{1}{\sin\theta}\frac{d}{d\phi}\left[J_{\frac{5}{2}}(k\alpha r)\, Y_{2,2}X_{2,2}\right.$$

$$\left. + J_{\frac{9}{2}}(k\alpha r)(Y_{4,2}P_{4,2}\cos 2\phi + Y_{4,4}P_{4,4}\cos 4\phi)\right]$$

$$+ \frac{1}{r\sin\theta}\frac{d^2}{d\phi\, dr}r^{\frac{1}{2}}\left\{\tfrac{1}{6}J_{\frac{5}{2}}(k\beta r)\, Z_{2,2}X_{2,2}\right.$$

$$\left. + \tfrac{1}{20}J_{\frac{9}{2}}(k\beta r)(Z_{4,2}P_{4,2}\cos 2\phi + Z_{4,4}P_{4,4}\cos 4\phi)\right]$$

$$- r^{-\frac{1}{2}}\frac{d}{d\theta}\left\{W_{3,2}P_{3,2}\sin 2\phi J_{\frac{7}{2}}(k\beta r)\right\}. \qquad (8)$$

§21. The several letters $Y$, $Z$, $W$ represent constants, the ratio between which will be determined from the surface conditions. The suffixes serve conveniently to show in the subsequent work the origin of the several terms.

Referring to equations (8) to (13) and (37) of Section I, and employing our previous notation, we find the surface conditions are as follows:

$$A_0Y_0 + \left\{1 + (\varepsilon P_2 + \varepsilon_2 X_{2,2})\, a\frac{d}{da}\right\}\left[(A_2Y_2 + B_2Z_2)P_2 + (A_2Y_{2,2} + B_2Z_{2,2})X_{2,2}\right]$$

$$- \left\{\varepsilon(C_2Y_{2,2} + D_2Z_{2,2}) + \varepsilon_2(C_2Y_2 + D_2Z_2)\right\}\frac{dP_2}{d\theta}\frac{dX_{2,2}}{d\theta}$$

$$- \varepsilon(C_2Y_2 + D_2Z_2)\left(\frac{dP_2}{d\theta}\right)^2$$

$$- \varepsilon_2(C_2Y_{2,2} + D_2Z_{2,2})\left\{\left(\frac{d}{d\theta}X_{2,2}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\right)^2\right\} + (A_4Y_4 + B_4Z_4)P_4$$

$$+ (A_4Y_{4,2} + B_4Z_{4,2})P_{4,2}\cos 2\phi + (A_4Y_{4,4} + B_4Z_{4,4})P_{4,4}\cos 4\phi = 0, \qquad (9)$$

$$\left\{1+(\epsilon P_2+\epsilon_2 X_{2,2})\,a\frac{d}{da}\right\}\left\{(C_2Y_2+D_2Z_2)\frac{dP_2}{d\theta}+(C_2Y_{2,2}+D_2Z_{2,2})\frac{dX_{2,2}}{d\theta}\right\}$$
$$-\frac{d}{d\theta}(\epsilon P_2+\epsilon_2 X_{2,2})\Big[(E_2Y_2+F_2Z_2)P_2+(E_2Y_{2,2}+F_2Z_{2,2})X_{2,2}$$
$$+\frac{d^2}{d\theta^2}(G_2Y_2+H_2Z_2)P_2+\frac{d^2}{d\theta^2}(G_2Y_{2,2}+H_2Z_{2,2})X_{2,2}\Big]$$
$$-\epsilon_2\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\frac{d^2}{d\theta d\phi}(G_2Y_{2,2}+H_2Z_{2,2})\operatorname{cosec}\theta X_{2,2}$$
$$+\frac{d}{d\theta}\left[(C_4Y_4+D_4Z_4)P_4+(C_4Y_{4,2}+D_4Z_{4,2})P_{4,2}\cos 2\phi\right.$$
$$\left.+(C_4Y_{4,4}+D_4Z_{4,4})P_{4,4}\cos 4\phi\right]$$
$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\{\tfrac{1}{2}aB_3W_{3,2}P_{3,2}\sin 2\phi\}=0, \tag{10}$$

$$\left\{1+(\epsilon P_2+\epsilon_2 X_{2,2})\,a\frac{d}{da}\right\}\frac{1}{\sin\theta}\frac{d}{d\phi}(C_2Y_{2,2}+D_2Z_{2,2})X_{2,2}$$
$$-\frac{d}{d\theta}(\epsilon P_2+\epsilon_2 X_{2,2})\frac{d^2}{d\theta d\phi}(G_2Y_{2,2}+H_2Z_{2,2})\operatorname{cosec}\theta X_{2,2}$$
$$-\epsilon_2\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\Big[(E_2Y_2+F_2Z_2)P_2+\cot\theta\frac{d}{d\theta}(G_2Y_2+H_2Z_2)P_2$$
$$+(E_2Y_{2,2}+F_2Z_{2,2})X_{2,2}$$
$$+\left(\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}+\cot\theta\frac{d}{d\theta}\right)(G_2Y_{2,2}+H_2Z_{2,2})X_{2,2}\Big]$$
$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\left[(C_4Y_{4,2}+D_4Z_{4,2})P_{4,2}\cos 2\phi+(C_4Y_{4,4}+D_4Z_{4,4})P_{4,4}\cos 4\phi\right]$$
$$-\frac{d}{d\theta}\{\tfrac{1}{2}aB_3W_{3,2}P_{3,2}\sin 2\phi\}. \tag{11}$$

In the first surface condition we have to express each term as a constant or a sum of spherical harmonics; in the second and third surface conditions each term must be expressed in terms of differential coefficients of spherical harmonics. The requisite results, if not all explicitly given on pp. 327–331, vol. 16 of this journal, are immediately deducible from the results there given. They consist of the expression of $(P_2)^2$, $P_2X_{2,2}$, $(X_{2,2})^2$, $\left(\frac{dP_2}{d\theta}\right)^2$, $\left(\frac{dX_{2,2}}{d\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\right)^2$, and

$\frac{dP_2}{d\theta}\frac{dX_{2,2}}{d\theta}$ as sums of constants and spherical harmonics; the expression of

$$P_2\frac{dP_2}{d\theta},\ P_2\frac{dX_{2,2}}{d\theta},\ X_{2,2}\frac{dP_2}{d\theta},\ X_{2,2}\frac{dX_{2,2}}{d\theta},\ \frac{dP_2}{d\theta}\frac{d^2P_{2,2}}{d\theta^2},\ \frac{dP_2}{d\theta}\frac{d^2X_{2,2}}{d\theta^2},\ \frac{dX_{2,2}}{d\theta}\frac{d^2P_2}{d\theta^2}$$

and $\frac{dX_{2,2}}{d\theta}\frac{d^2X_{2,2}}{d\theta^2}+\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})$ as differential coefficients with respect to $\theta$ of a sum of spherical harmonics; the expression of

$$P_2\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi},\ \cot\theta\frac{dP_2}{d\theta}\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi},\ X_{2,2}\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi},\ \frac{dP_2}{d\theta}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})$$

and $\frac{dX_{2,2}}{d\theta}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})+\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2}+\cot\theta\frac{d}{d\theta}\right)X_{2,2}$

as differentials with respect to $\phi$ of a sum of spherical harmonics divided by $\sin\theta$.

It appears unnecessary to reproduce the results.

Use must also be made of the identities

$$\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{3,2}\sin 2\phi)=\frac{d}{d\theta}(P_{3,2}\cos 2\phi),$$

$$\frac{d}{d\theta}(P_{3,2}\sin 2\phi)=\frac{1}{\sin\theta}\frac{d}{d\phi}(\tfrac{4}{7}P_{2,2}-\tfrac{3}{7}P_{4,2})\cos 2\phi.$$

§22. Exactly the same method of procedure is to be followed as in the case of equilibrium, so it will suffice to record the resulting equations. These are as follows:

$$Y_0=-\frac{1}{5A_0}a\frac{d}{da}\left[\epsilon(A_2Y_2+B_2Z_2)+\tfrac{3}{7}\epsilon_2(A_2Y_{2,2}+B_2Z_{2,2})\right], \quad (12)$$

$$\left.\begin{aligned}A_2Y_2+B_2Z_2&=-\tfrac{2}{7}a\frac{d}{da}\left[\epsilon(A_2Y_2+B_2Z_2)-\tfrac{3}{2}\epsilon_2(A_2Y_{2,2}+B_2Z_{2,2})\right],\\ C_2Y_2+D_2Z_2&=-\tfrac{1}{7}\Big[\epsilon\Big\{a\frac{d}{da}(C_2Y_2+D_2Z_2)-(E_2Y_2+F_2Z_2)\\ &\quad-3(G_2Y_2+H_2Z_2)\Big\}-\tfrac{3}{2}\epsilon_2\Big\{a\frac{d}{da}(C_2Y_{2,2}+D_2Z_{2,2})\\ &\quad-(E_2Y_{2,2}+F_2Z_{2,2})-3(G_2Y_{2,2}+H_2Z_{2,2})\Big\}\Big],\end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned}
A_2 Y_{2,2} + B_2 Z_{2,2} = {} & \tfrac{2}{7}\, a \frac{d}{da} \left[ \varepsilon \left( A_2 Y_{2,2} + B_2 Z_{2,2} \right) + \varepsilon_2 \left( A_2 Y_2 + B_2 Z_2 \right) \right], \\
C_2 Y_{2,2} + D_2 Z_{2,2} = {} & \tfrac{1}{7} \Big[ \varepsilon \Big\{ a \frac{d}{da} \left( C_2 Y_{2,2} + D_2 Z_{2,2} \right) \\
& - \left( E_2 Y_{2,2} + F_2 Z_{2,2} \right) - 3 \left( G_2 Y_{2,2} + H_2 Z_{2,2} \right) \Big\} \\
& + \varepsilon_2 \Big\{ a \frac{d}{da} \left( C_2 Y_2 + D_2 Z_2 \right) - \left( E_2 Y_2 + F_2 Z_2 \right) \\
& - 3 \left( G_2 Y_2 + H_2 Z_2 \right) \Big\} \Big],
\end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned}
A_4 Y_4 + B_4 Z_4 = {} & - \tfrac{2}{35}\, a \frac{d}{da} \left[ 9\varepsilon \left( A_2 Y_2 + B_2 Z_2 \right) + 2\varepsilon_2 \left( A_2 Y_{2,2} + B_2 Z_{2,2} \right) \right], \\
C_4 Y_4 + D_4 Z_4 = {} & - \tfrac{1}{35} \Big[ 9\varepsilon \Big\{ a \frac{d}{da} \left( C_2 Y_2 + D_2 Z_2 \right) \\
& - \left( E_2 Y_2 + F_2 Z_2 \right) + 4 \left( G_2 Y_2 + H_2 Z_2 \right) \Big\} \\
& + 2\varepsilon_2 \Big\{ a \frac{d}{da} \left( C_2 Y_{2,2} + D_2 Z_{2,2} \right) \\
& - \left( E_2 Y_{2,2} + F_2 Z_{2,2} \right) + 4 \left( G_2 Y_{2,2} + H_2 Z_{2,2} \right) \Big\} \Big],
\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}
A_4 Y_{4,2} + B_4 Z_{4,2} = {} & - \tfrac{3}{2}\, a \frac{d}{da} \left[ \varepsilon \left( A_2 Y_{2,2} + B_2 Z_{2,2} \right) + \varepsilon_2 \left( A_2 Y_2 + B_2 Z_2 \right) \right], \\
C_4 Y_{4,2} + D_4 Z_{4,2} = {} & - \tfrac{3}{4} \Big[ \varepsilon \Big\{ a \frac{d}{da} \left( C_2 Y_{2,2} + D_2 Z_{2,2} \right) \\
& - \left( E_2 Y_{2,2} + F_2 Z_{2,2} \right) + 4 \left( G_2 Y_{2,2} + H_2 Z_{2,2} \right) \Big\} \\
& + \varepsilon_2 \Big\{ a \frac{d}{da} \left( C_2 Y_2 + D_2 Z_2 \right) \\
& - \left( E_2 Y_2 + F_2 Z_2 \right) + 4 \left( G_2 Y_2 + H_2 Z_2 \right) \Big\} \Big],
\end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned}
A_4 Y_{4,4} + B_4 Z_{4,4} = {} & - \tfrac{1}{2}\, \varepsilon_2 a \frac{d}{da} \left( A_2 Y_{2,2} + B_2 Z_{2,2} \right), \\
C_4 Y_{4,4} + D_4 Z_{4,4} = {} & - \tfrac{1}{4}\, \varepsilon_2 \Big\{ a \frac{d}{da} \left( C_2 Y_{2,2} + D_2 Z_{2,2} \right) \\
& - \left( E_2 Y_{2,2} + F_2 Z_{2,2} \right) + 4 \left( G_2 Y_{2,2} + H_2 Z_{2,2} \right) \Big\},
\end{aligned}\right\} \quad (17)$$

$$\begin{aligned}
W_{3,2} = \frac{1}{aB_3} \Big[ & - \varepsilon \Big\{ a \frac{d}{da} \left( C_2 Y_{2,2} + D_2 Z_{2,2} \right) + E_2 Y_{2,2} + F_2 Z_{2,2} \Big\} \\
& + \varepsilon_2 \Big\{ a \frac{d}{da} \left( C_2 Y_2 + D_2 Z_2 \right) + \left( E_2 Y_2 + F_2 Z_2 \right) \Big\} \Big]. \quad (18)
\end{aligned}$$

The three surface conditions are satisfied exactly, to the assumed degree of approximation, when the equations (12) to (18) are satisfied.

As first approximations, when terms in $\varepsilon$ and $\varepsilon_2$ are neglected, we obtain from (13) and (14)

$$\left.\begin{aligned} A_2 Y_2 + B_2 Z_2 &= 0, \\ C_2 Y_2 + D_2 Z_2 &= 0, \end{aligned}\right\} \qquad (19)$$

$$\left.\begin{aligned} A_2 Y_{2,2} + B_2 Z_{2,2} &= 0, \\ C_2 Y_{2,2} + D_2 Z_{2,2} &= 0. \end{aligned}\right\} \qquad (20)$$

Eliminating $Y_2/Z_2$ from (19), and $Y_{2,2}/Z_{2,2}$ from (20), we arrive in either case at the frequency equation,

$$A_2 D_2 - B_2 C_2 = 0. \qquad (21)$$

This supplies, of course, the first approximation to the values of $k$ in the present problem.

In a perfect sphere the constants $Y_2$, $Y_{2,2}$ are absolutely independent of one another and their ratio may have any value. The vibrations answering to $P_2$ and those answering to $X_{2,2}$, though possessing the same frequency, may exist side by side or separately, uninfluenced the one by the other.

§23. The same independence is exhibited when the ellipsoid (3) is a spheroid. For putting $\varepsilon_2 = 0$ in (13) and (14) we get two completely independent sets of equations, viz.

$$\left.\begin{aligned} A_2 Y_2 + B_2 Z_2 &= -\tfrac{2}{7}\varepsilon a \frac{d}{da}(A_2 Y_2 + B_2 Z_2), \\ C_2 Y_2 + D_2 Z_2 &= -\tfrac{1}{7}\varepsilon \Big[ a \frac{d}{da}(C_2 Y_2 + D_2 Z_2) \\ &\qquad - (E_2 Y_2 + F_2 Z_2) - 3(G_2 Y_2 + H_2 Z_2) \Big] \end{aligned}\right\} \qquad (22)$$

and

$$\left.\begin{aligned} A_2 Y_{2,2} + B_2 Z_{2,2} &= \tfrac{2}{7}\varepsilon a \frac{d}{da}(A_2 Y_{2,2} + B_2 Z_{2,2}), \\ C_2 Y_{2,2} + D_2 Z_{2,2} &= \tfrac{1}{7}\varepsilon \Big[ a \frac{d}{da}(C_2 Y_{2,2} + D_2 Z_{2,2}) \\ &\qquad - (E_2 Y_{2,2} + F_2 Z_{2,2}) - 3(G_2 Y_{2,2} + H_2 Z_{2,2}) \Big]. \end{aligned}\right\} \qquad (23)$$

Proceeding in the same manner as in previous cases, we find for the increments of pitch in the spheroid, relative to the pitch in the perfect sphere, from

(22) and (23) respectively

$$\delta k_1/k = -N\varepsilon, \tag{24}$$

$$\delta k_2/k = +N\varepsilon, \tag{25}$$

where

$$N = \tfrac{1}{4}\left[2 - \frac{A_2 a\dfrac{dD_2}{da} - B_2 a\dfrac{dC_2}{da} + (A_2F_2 - B_2E_2) + 3(A_2H_2 - B_2G_2)}{a\dfrac{d}{da}(A_2D_2 - B_2C_2)}\right]. \tag{26}$$

The identity of $-N$ with the coefficient of $\varepsilon_2$ in (14), Section V, when 2 is written for $i$, should be noticed.

It will be observed that $\delta k_1$ refers to the vibration depending on $P_2$, $\delta k_2$ to that depending on $X_{2,2}$.

§24. In the general case of the ellipsoid, the independence of the vibrations depending on $P_2$ and on $X_{2,2}$ ceases. For equations (13) and (14) lead to

$$Y_{2,2}/Y_2 = Z_{2,2}/Z_2 = q, \tag{27}$$

where

$$q\varepsilon + \varepsilon_2 = q(-\varepsilon + \tfrac{1}{3}q\varepsilon_2), \tag{28}$$

or

$$q = \tfrac{3}{2}\left\{\frac{\varepsilon}{\varepsilon_2} \mp \sqrt{\left(\frac{\varepsilon}{\varepsilon_2}\right)^2 + \tfrac{1}{3}}\right\}. \tag{29}$$

Employing (27) in either (13) or (14), we get

$$\left.\begin{aligned} A_2Y_2 + B_2Z_2 &= \tfrac{1}{4}\frac{q\varepsilon + \varepsilon_2}{q}\, a\frac{d}{da}(A_2Y_2 + B_2Z_2), \\ C_2Y_2 + D_2Z_2 &= \tfrac{1}{4}\frac{q\varepsilon + \varepsilon_2}{q}\Big[a\frac{d}{da}(C_2Y_2 + D_2Z_2) \\ &\qquad - (E_2Y_2 + F_2Z_2) - 3(G_2Y_2 + H_2Z_2)\Big]. \end{aligned}\right\} \tag{30}$$

Comparing (30) with (23), we deduce at once for the change of frequency due to the departure of the surface from the truly spherical form

$$\begin{aligned} \delta k/k &= N(q\varepsilon + \varepsilon_2)/q \\ &= \mp N\sqrt{\varepsilon^2 + \tfrac{1}{3}\varepsilon_2^2} \end{aligned} \tag{31}$$

by (29), where $N$ is given by (26).

The interpretation to be put upon the result is that in the general case of the ellipsoid (3), the principal terms in $P_2$ and $X_{2,2}$ do not occur independently but stand to one another in one of two definite ratios given by the values of $q$ in

(29). In one of the cases the pitch is raised relative to what it would be in the perfect sphere, in the other it is lowered by an equal amount.

By means of (4) we may exhibit (31) in the more symmetrical form

$$\frac{\delta k}{k} = \mp 2N\{1 - 3(a'b' + a'c' + b'c')/(a' + b' + c')^2\}^{\frac{1}{2}}. \tag{32}$$

§25. The values of the constants appearing in the subsidiary terms are derived from (12) and (15) to (18), use being made of (19), (20) and (21).

For shortness let

$$\mu_2 = \frac{(Y_2/B_2)}{140(A_4D_4 - B_4C_4)}\Big[-B_4\Big\{A_2a\frac{dD_2}{da} - B_2a\frac{dC_2}{da} - (A_2F_2 - B_2E_2) + 4(A_2H_2 - B_2G_2)\Big\} + 2D_4\Big(A_2a\frac{dB_2}{da} - B_2a\frac{dA_2}{da}\Big)\Big], \tag{33}$$

$$\mu_2' = \frac{(Y_2/B_2)}{140(A_4D_4 - B_4C_4)}\Big[A_4\Big\{A_2a\frac{dD_2}{da} - B_2a\frac{dC_2}{da} - (A_2F_2 - B_2E_2) + 4(A_2H_2 - B_2G_2)\Big\} - 2C_4\Big(A_2a\frac{dB_2}{da} - B_2a\frac{dA_2}{da}\Big)\Big]. \tag{34}$$

Then we get

$$Y_0 = -\frac{(\epsilon + \frac{4}{3}q\epsilon_2)Y_2}{5A_0B_2}\Big(B_2a\frac{dA_2}{da} - A_2a\frac{dB_2}{da}\Big), \tag{35}$$

$$W_{8,2} = \frac{(q\epsilon - \epsilon_2)Y_2}{aB_2B_8}\Big\{A_2a\frac{dD_2}{da} - B_2a\frac{dC_2}{da} + (A_2F_2 - B_2E_2)\Big\}, \tag{36}$$

$$Y_4/\mu_2 = Z_4/\mu_2' \quad = 36\epsilon + 8q\epsilon_2, \tag{37}$$

$$Y_{4,2}/\mu_2 = Z_{4,2}/\mu_2' = 105(q\epsilon + \epsilon_2), \tag{38}$$

$$Y_{4,4}/\mu_2 = Z_{4,4}/\mu_2' = 35q\epsilon_2. \tag{39}$$

To simplify the results, we may make use of (21) and of the results (18), (19), etc., of Section I. We in this way reduce (35) and (36) to

$$Y_0 = -\frac{(\epsilon + \frac{4}{3}q\epsilon_2)C_2Y_2}{5A_0}\,\frac{(3y^2 - 4x^2)(2 + A_2/C_2) - \frac{1}{2}y^4}{y^2 - 8}, \tag{40}$$

$$W_{8,2} = -\frac{(q\epsilon - \epsilon_2)A_2Y_2}{12aB_8}\,\frac{y^2(y^2 - 10)}{y^2 - 8}. \tag{41}$$

Referring to (29) it will be found that $Y_0$ vanishes when either

$$\epsilon_2 = 0 \text{ and } Y_2 = 0,$$

or when

$$2\epsilon_2 \pm 3\epsilon = 0, \tag{42}$$

and the proper sign is taken in the value of $q$.

The first alternative answers to a spheroid whose axis is $\theta = 0$, the vibrations which require no subsidiary pure radial term being those which depend on the harmonic $P_{2,2} \cos 2\phi$.

The equation (42) is satisfied when

$$c' = b'$$

or

$$c' = a';$$

and so likewise implies the ellipsoid being a surface of revolution. When $c' = b'$ we have

$$q = \tfrac{1}{2},$$

and the harmonic occurring in the displacements is

$$P_2 + \tfrac{1}{2} P_{22} \cos 2\phi \equiv \cos^2\theta - \sin^2\theta \sin^2\phi;$$

when $c' = a'$ we have

$$q = -\tfrac{1}{2},$$

and the harmonic occurring in the displacements is

$$P_2 - \tfrac{1}{2} P_{22} \cos 2\phi \equiv \cos^2\theta - \sin^2\theta \cos^2\phi.$$

If for a moment we suppose rectangular coordinates $x$, $y$, $z$ coinciding with the principal axes of the ellipsoid, then all three cases are included in the following statement: If $z = 0$ be an axis of symmetry, then no subsidiary pure radial displacement is required if the surface harmonic in the principal term of the radial component of the displacement is $(x^2 - y^2)/r^2$.

In a somewhat similar fashion it is easy to prove from (41) and (29) that $W_{3,2}$ vanishes when the ellipsoid is one of revolution, the harmonic appearing in the principal term of the radial displacement having in this case the axis of symmetry for its axis; if, for instance, $\theta = 0$ is the axis of symmetry, the harmonic is $P_2$.

The close analogy of the results (37), (38), (39) to the corresponding results for the equilibrium problem—vol. 16, equations (263), (266) and (270), p. 377—is noteworthy. In fact, putting

$$\epsilon_1 = \alpha_2 = \beta_2 = 0,$$
$$\mu_{22}/\mu_2 = \mu'_{22}/\mu'_2 = q,$$

in the equilibrium results, we make them identical in form with (37), (38) and (39).

# *Non-Uniform Convergence and the Integration of Series Term by Term.*

Presented to the American Mathematical Society, 31 August 1896.

By William F. Osgood.

The subject of this paper is the study (I) of the manner of the convergence of a function $s_n(x)$, satisfying Conditions $(A)$, when $n$ becomes infinite (v. §§1, 2); and (II) of the conditions under which

$$\int_{x_0}^{x} \lim_{n=\infty} s_n(x)\,dx = \lim_{n=\infty} \int_{x_0}^{x} s_n(x)\,dx$$

(v. §13). The principal results are stated in the italicized theorems and paragraphs, and a table of contents is added, chiefly to enable the reader to inform himself more readily concerning the nomenclature.

The four problems of

1) integration of a series term by term,
2) differentiation of a series term by term,
3) reversal of the order of integration in a double-integral,
4) differentiation under the sign of integration,

are in certain classes of cases but different forms of the same problem, a problem in double limits; so that a theorem applying to one of these problems yields at once a theorem applying to the other three. It did not seem best to go into this question in this paper, but one such theorem is stated as an example in the last paragraph.

## I.—Non-Uniform Convergence.

1. 1) Let $u_1(x)$, $u_2(x)$, .... be a set of single-valued real functions of the real variable $x$, which for all values of $x$ in the finite interval $L: a \leq x \leq b$ are continuous. Form the sum:

$$s_n(x) = u_1(x) + u_2(x) + \dots + u_n(x),$$

then $s_n(x)$ is also continuous in $L$.*

* In this paper an interval is to be understood as including among its points its extremities.

2) If a point $x$ of $L$ be chosen at pleasure and then regarded as fixed, $s_n(x)$ shall converge toward a limit when $n$ increases indefinitely. Denote this limit by $f(x)$:

$$f(x) = \lim_{n=\infty} s_n(x) = u_1(x) + u_2(x) + \dots .$$

3) Finally, $f(x)$ shall be a continuous function of $x$.

*Conditions* $(A)$.—This set of conditions will be referred to in the following pages as *Conditions* $(A)$.

We propose to investigate the most general manner of the convergence of $s_n(x)$ toward its limit $f(x)$ as $n$ increases indefinitely, *when no further restrictions are placed on* $s_n(x)$ *than Conditions* $(A)$.

2. In general the convergence will be non-uniform. Examples of such convergence are now familiar in *arithmetic* form. Thus:

*Ex.* 1. $$s_n(x) = \frac{n^2 x}{1 + n^3 x^2}, \qquad f(x) = 0, \qquad 0 \leq x \leq 1.$$

The geometric representation of the non-uniform convergence by means of the approximation curves

$$y = s_n(x)$$

is given in Fig. 1. The peaks rise higher and higher as $n$ increases and their

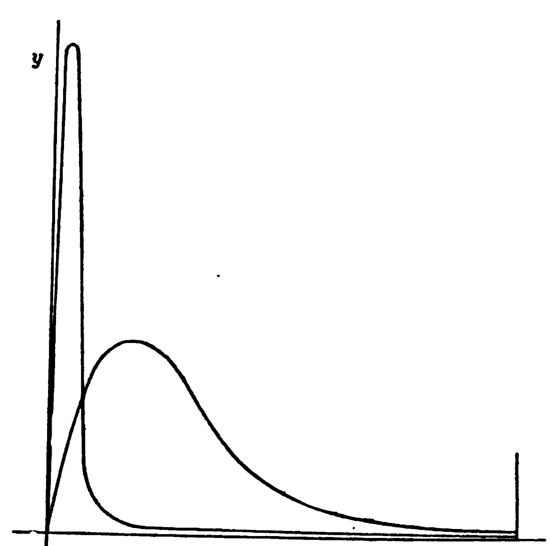


FIG. 1.

altitude transcends ultimately any magnitude $m$ chosen at pleasure, no matter how large. I will express this in the form: $s_n(x)$ has infinite peaks in the neighborhood of the point $x = 0$.

*Definition of X-Points.*—A point $x_0$ in whose neighborhood $s_n(x)$ has infinite peaks shall be denoted by $X$ and designated as a $X$-point.

*Ex.* 2. $$s_n(x) = \frac{nx}{1 + n^2x^2}, \qquad f(x) = 0, \qquad 0 \leqq x \leqq 1.$$

Here the altitude of each peak is $\frac{1}{2}$ (Fig. 2).

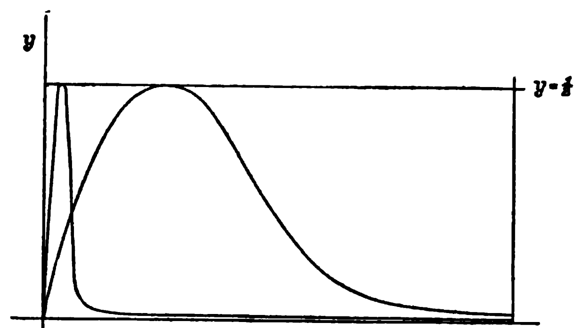


FIG. 2.

In the above examples there is only *one* point in the interval $0 \leqq x \leqq 1$ in whose neighborhood peaks occur, namely the point $x = 0$. And if any interval not abutting on the point $x = 0$ be picked out from the above interval, $s_n(x)$ will, in it, converge uniformly toward its limit. But it is easy to see that such points can occur frequently. Thus if in either of the above examples $x$ be replaced by $\sin^2 k\pi x$, where $k$ is a constant positive integer, peaks will occur in the neighborhood of each of the $(k + 1)$ points (Fig. 3),

$$0, \ 1/k, \ 2/k, \ldots . k/k = 1.$$

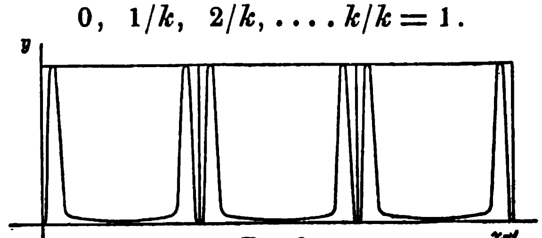


FIG. 3.

*Ex.* 3. In fact, by the aid of Ex. 2, a function $s_n(x)$ can be readily formed such that, if $x_0$ be any rational value of $x$ whatsoever, peaks will occur in the neighborhood of $x_0$. Let

$$\phi_k(x) = \frac{n \sin^2 k\pi x}{1 + n^2 \sin^4 k\pi x},$$

and form the series:

$$s_n(x) = \phi_{1!}(x) + \frac{1}{2!}\phi_{2!}(x) + \frac{1}{3!}\phi_{3!}(x) + \ldots . = \sum_{i=1}^{\infty}\frac{1}{i!}\phi_{i!}(x).$$

This series, regarded as a function of the two independent variables $n, x$, converges uniformly. Hence for a given value of $n$, since each of the terms is a continuous function of $x$, the series defines a continuous function of $x$; while for a given value of $x$

$$\lim_{n=\infty} s_n(x) = \lim_{n=\infty}\phi_{1!}(x) + \lim_{n=\infty}\frac{1}{2!}\phi_{2!}(x) + \ldots . = 0.$$

It is easy to see how the approximation curves

$$y = s_n(x)$$

look. Think of $n$ as large. Then the earlier terms in the $s_n(x)$ series will, individually, appear as indicated in Fig. 4 (these curves are all similar to each

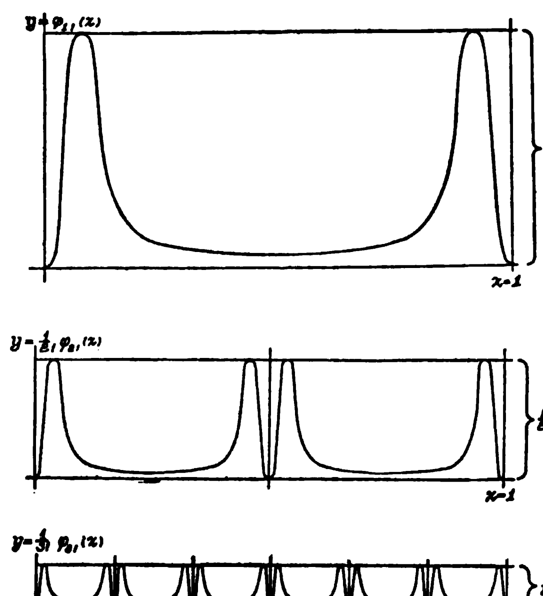

FIG. 4.

other), while the later terms, their maximum ordinates being very small, yield but a small contribution to the sum. Thus in the neighborhood of $x = \frac{1}{2}$ the peaks that come from $\frac{1}{2!}\phi_{2!}(x)$, their height being $\frac{1}{2}\cdot\frac{1}{2!}$, are predominant in determining the character of $s_n(x)$ in the neighborhood of this point; for all the subsequent terms together cannot for any value of $x$ yield a sum greater than

$$\frac{1}{2}\cdot\frac{1}{3!}+\frac{1}{2}\cdot\frac{1}{4!}+\ldots.<\frac{1}{2}\;\frac{1}{3!}\;\frac{4}{3}<\frac{1}{2}\cdot\frac{1}{4}$$

or less than 0, while the contribution of the preceding terms $(\phi_{1!}(x))$ is small in the neighborhood of $x=\frac{1}{2}$. And so it is in the general case. Let $x=\frac{p}{q}$ be any positive fraction less than unity, in reduced form. Let $k$ be the smallest integer such that $k!$ is divisible by $q$. Then in the neighborhood of the point $x=\frac{p}{q}$, the peaks that come from the term $\frac{1}{k!}\phi_{k!}(x)$ will be predominant in determining the character of $s_n(x)$ in the neighborhood of this point. Hence this function converges non-uniformly toward its limit *in every interval** that can be chosen from the given interval $0\leq x\leq 1$. The upper limit (resp. maximum) of the heights of the peaks is not, however, the same in all intervals; in fact, in some intervals it is very small. We may think of this upper limit as measuring, so to speak, the *strength* of the non-uniform convergence and say: the non-uniformity of the convergence is stronger in some intervals than in others.

3. We turn now to the general case and prove a fundamental theorem concerning the manner of the convergence of $s_n(x)$ toward its limit.

That which is essential in this matter will come out more clearly if we study, not the function $s_n(x)$, but

$$S_n(x)=s_n(x)-f(x),$$

the limit of which for a given value of $x$ and $n=\infty$ is 0. This substitution amounts arithmetically to subtracting $f(x)$ from the first term of the series, $u_1(x)$; geometrically, to dropping the curve $f(x)$ down on the $x$-axis, as indicated

* The method by which this function has been formed is essentially the same as Hankel's Princip der Verdichtung der Singularitäten. Cf. Hankel, Unendlich oft oscillirende und unstetige Functionen, Tübingen, 1870; Dini, Fondamenti per la teorica delle funzioni di variabili reali, Pisa, 1878; German translation by Lüroth and Schepp, Leipzig, 1892.

in Fig. 5, the approximation curves $y = s_n(x)$ having each of their ordinates

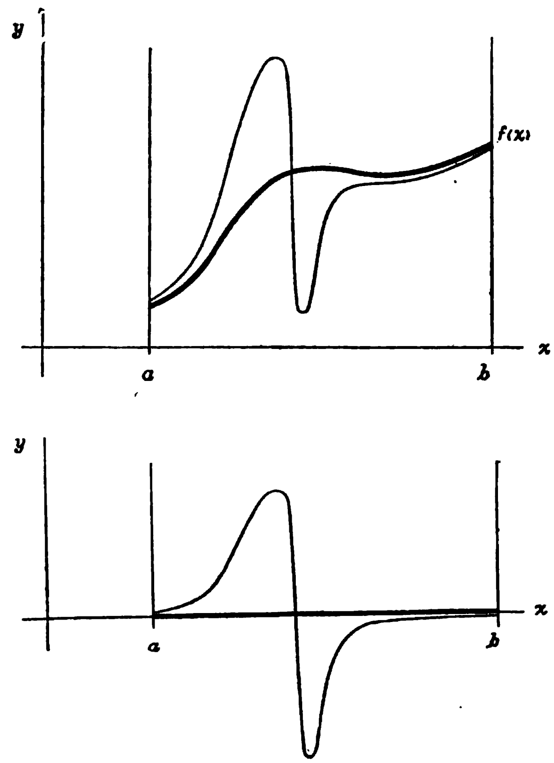


FIG. 5.

changed by the same amount. $S_n(x)$, like $s_n(x)$, satisfies Conditions ($A$) and further,

$$\lim_{n=\infty} S_n(x) = 0.$$

Let $A$ be a positive quantity chosen at pleasure and let $x_0$ be any point of $L$ (Fig. 6). Since $\lim_{n=\infty} S_n(x_0) = 0$, there exists a fixed integer $m$ such that

$$|S_n(x_0)| < A, \qquad n \geqq m.$$

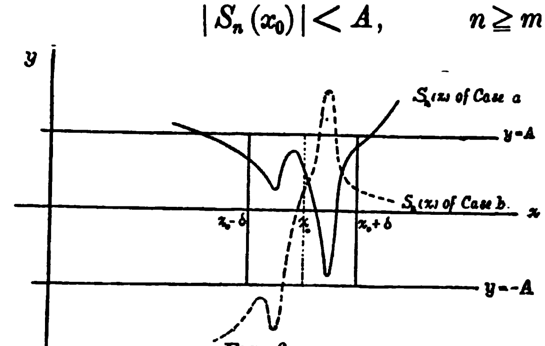


FIG. 6.

Consider the behavior of $S_n(x)$ in the neighborhood of $x_0$ when $n$ increases indefinitely. One of two cases must arise; either $a$) it is possible to name a positive quantity $\delta$ and a positive integer $m$ such that for all values of $x$ lying in the interval $(x_0 - \delta,\ x_0 + \delta)$

$$|S_n(x)| < A, \qquad n \geqq m;$$

or $b$) no matter how small $\delta$ and how large $m$ be taken (and then held fast), a value of $n$ greater than $m$ can then be chosen so that

$$|S_n(x)| \geqslant A$$

somewhere in the interval $(x_0 - \delta,\ x_0 + \delta)$.

*Definition of $\gamma$-Points.* In Case $b$) $x_0$ shall be denoted by $\gamma$ and designated as a $\gamma$-point.

Such points depend in general on the choice of $A$. If a new value $A' < A$ be taken, all the points $\gamma_A$ that formerly were $\gamma$-points remain such, but new points may claim membership in the $\gamma_{A'}$-list.

*Conditions* ($P$). A set of points* which 1) is nowhere dense and 2) contains its derivative shall be said to satisfy *Conditions* ($P$). Such a set will usually be denoted by the letter $G$, to which suffixes may be attached. An example typical for the most general set $G$ is given in §8.

FUNDAMENTAL THEOREM.—*Let the positive quantity $A$ be chosen at pleasure. Then the corresponding $\gamma$-points form a set of points $G$ satisfying Conditions* ($P$).

The proof is as follows. That $G$ contains its derivative is evident from the nature of the $\gamma$-points. Suppose that throughout a certain sub-interval $(\alpha, \beta)$ of

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*No technical knowledge of the theory of Sets of Points (Punktmengen) will be assumed in this paper (except in §21, where explicit references are given). But the conceptions involved in the definitions to be given presently will be necessary for an understanding of what follows. Cf. G. Cantor's early papers in Crelle and the Math. Ann., translated into French and reprinted in the Acta Math., vol. 2, 1883; also Dini, Ch. II.

I will translate *Punktmenge* = *ensemble* by *set of points; überall dicht* = *condensé* by *dense; Ableitung* = *dérivé* by *derivative; abzählbar* = *dénombrable* by *enumerable.*

A set of points is said to be *dense throughout an interval* if, $x_0$ being any point lying within the interval, it is impossible to enclose $x_0$ within an interval $(x_0 - \delta,\ x_0 + \delta)$ which is free from points of the set. If an interval $L$ contains no sub-interval whatsoever throughout which the given set is dense, then the set is said to be *nowhere dense in L*. By the *derivative* of a set of points is meant that set whose points are each the limit approached by some sub-set of points of the original set. Thus each point of the derivative is a *limiting point* (*Häufungsstelle*) of the original set. A set of points is said to *contain its derivative* if the points of the derivative set all belong to the given set. derivative = closure

$L$ the $\gamma$-points were dense (Fig. 7). Let $c$ be any $\gamma$-point of this interval except-

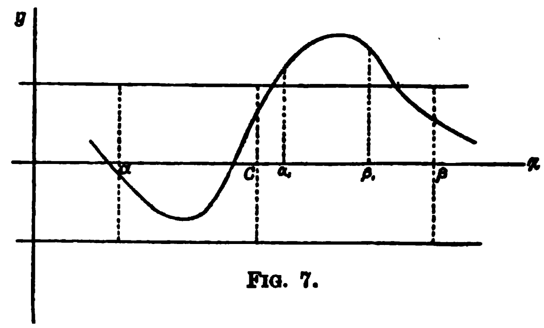


FIG. 7.

ing an extremity: $\alpha < c < \beta$. Then $n_1$ can be so determined that in portions of at least one of the intervals $(\alpha, c)$, $(c, \beta)$

$$|S_{n_1}(x)| > A.$$

It will thus be possible to separate from $(\alpha, \beta)$ a first sub-interval $(\alpha_1, \beta_1)$ such that

$$|S_{n_1}(x)| > A, \qquad \alpha_1 \leqq x \leqq \beta_1.$$

Next proceed with the sub-interval $(\alpha_1, \beta_1)$ in the same way as originally $(\alpha, \beta)$ was treated. In this way an $S_{n_2}(x)$, $n_2 > n_1$, and a sub-interval $(\alpha_2, \beta_2)$ are found such that $\alpha_1 \leqq \alpha_2$, $\beta_2 \leqq \beta_1$ and

$$|S_{n_2}(x)| > A, \qquad \alpha_2 \leqq x \leqq \beta_2.$$

The repetition of this step leads to a series of functions $S_{n_i}(x)$ and a series of intervals $(\alpha_i, \beta_i)$, where $\alpha_i < \beta_i$, $\alpha \leqq \alpha_1 \leqq \alpha_2 \ldots.$, $\beta \geqq \beta_1 \geqq \beta_2 \ldots.$, and

$$|S_{n_i}(x)| > A, \qquad \alpha_i \leqq x \leqq \beta_i.$$

Thus the $\alpha$'s and the $\beta$'s converge toward limits, and they shall be so chosen that these limits are equal:

$$\lim_{i=\infty} \alpha_i = \lim_{i=\infty} \beta_i = \bar{c}.$$

Then is $$\alpha_i < \bar{c} < \beta_i.$$

Hence $$|S_{n_j}(\bar{c})| > A, \qquad j = 1, 2, \ldots.$$

But this is impossible, since $\lim_{n=\infty} S_n(\bar{c}) = 0$, and thus the theorem is established.

4. By the aid of the Fundamental Theorem we will now study the manner of the convergence of $s_n(x)$ toward its limit $f(x)$.

THEOREM I.—*The X-points of the interval L form* AT MOST *a set G satisfying Conditions* $(P)$.*

From the nature of the $X$-points it is evident that $G$ contains its derivative. Furthermore these $X$-points will also be $X$-points of $S_n(x)$. Let the positive quantity $A$ be taken at pleasure. The corresponding $\gamma_A$-points will by the Fundamental Theorem form a set satisfying Conditions $(P)$ and will in general be composed of two classes of $\gamma$-points: *a*) those $\gamma$-points which drop out (cease to be $\gamma$-points) when the value of $A$ is increased; and *b*) other points that remain $\gamma$-points, no matter how large $A$ be taken. These latter are the $X$-points, and since they are at most a part of a set nowhere dense, they must themselves form such a set. Moreover this set contains its derivative. The $X$-points form therefore a set satisfying Conditions $(P)$.

5. It thus appears that if $x_0$ is any point of the interval $L$, there will be in every neighborhood of $x_0$ an interval which is entirely free from $X$-points. To such an interval $(\alpha, \beta)$ there corresponds a fixed number $B$, different for different intervals, such that

$$|S_n(x)| < B, \qquad \alpha \leqq x \leqq \beta, \qquad n = 1, 2, \ldots .$$

For suppose this were not the case. Then there would exist in $(\alpha, \beta)$ an $x_1$ for

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* Du Bois Reymond has given in the Sitz.-Ber. d. Berliner Akad., 1886, p. 359, an example of a function which satisfies, as he believed, Conditions (A), while its $X$-points are dense throughout an interval, and hence from their nature fill the interval. His proof turns ultimately on the limits approached by certain complicated expressions, and at this part of his paper he restricts himself to assertions (top of p. 370). But at an earlier point of the paper (p. 368, line 15) he says, referring to Kronecker's sufficient condition that a series may be integrated term by term (ibid. 1878, p. 54): "Aber diese ausreichende Bedingung ist offenbar (sic) notwendig, mithin ist sie mit der Forderung $Q(x) = 0$ [in the notation of this paper, $Q(x) = \lim_{n=\infty} \int_{x_0}^{x_1} S_n(x)\, dx$] vollständig aequivalent." From this and subsequent passages it appears that he overlooked the possibility of the $X$-points resp. the $\gamma$-points forming the most general set of points satisfying Conditions (P).

which

$$|S_{n_1}(x_1)| > 1;$$

an $x_2$ for which

$$|S_{n_2}(x_2)| > 2, \qquad n_2 > n_1,$$

etc.; generally, an $x_i$ for which

$$|S_{n_i}(x_i)| > i, \qquad n_i > n_{i-1}.$$

The set of points $x_1, x_2, \ldots.$ must have at least one limiting point $\bar{x}$ belonging to the interval $(\alpha, \beta)$, for at most a finite number of the $x_i$'s can coincide. But $\bar{x}$ would then be a $X$-point, and thus the above assertion is seen to be true.

From among all the quantities that can serve in the capacity of the above $B$'s, let that one be picked out which is the lower limit of the $B$'s and let it be denoted, for it is also a $B$, by $B'$. Then

$$|S_n(x)| \leqq B', \qquad \alpha \leqq x \leqq \beta, \qquad n = 1, 2, \ldots.,$$

while

$$|S_n(x)| > B' - \varepsilon$$

for some values of $n$, $x$, if $\varepsilon > 0$.

Hitherto $n$ has begun with the value 1. If now it begin with the value $m \geqq 1$: $n = m, m+1, \ldots.$, the corresponding $B'$, which shall be denoted by $B'_m$, will never increase with $m$, but it may decrease. Let

$$\lim_{m=\infty} B'_m = \bar{B}.$$

*Coefficient of Convergence.* We have already spoken of the *strength* of the non-uniformity of the convergence in an interval (§2, end). It is desirable to make this conception precise by defining the magnitude $\bar{B}$ as the *Coefficient of the Convergence in the Interval* $(\alpha, \beta)$ and to take this coefficient as measuring the strength of the non-uniformity of the convergence. Its value will then characterize the quality of the convergence in the interval as a whole. Thus when $\bar{B} = 0$ the convergence will be uniform, and conversely.

A simple example to illustrate the foregoing will not be out of place. Let

$$s_n(x) = \frac{n+1}{n} \cdot \frac{nx}{1+n^2x^2} = S_n(x).$$

$S_n(x)$ has only one peak (Fig. 8) and its altitude is $\frac{n+1}{n} \cdot \frac{1}{2}$. Hence $B' = 1$; $B'_m = \left(1 + \frac{1}{m}\right) \cdot \frac{1}{2}$; $\bar{B} = \frac{1}{2}$.

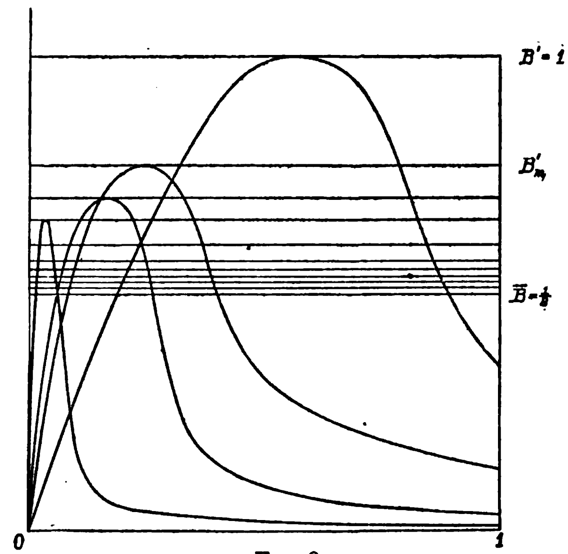


FIG. 8.

6. In the preceding paragraph a necessary condition has been established for the set of points in whose neighborhood $s_n(x)$ has infinite peaks (the $X$-points). In the present paragraph a necessary condition shall be established for the set of points in whose neighborhood $s_n(x)$ has null peaks (the $\zeta$-points).

*Definition of $\zeta$-Points.* If a point $x_0$ is such that, after the arbitrary choice of a positive quantity $\varepsilon$, an interval $(x_0 - \delta,\ x_0 + \delta)$ can be determined whose coefficient of convergence $\bar{B}$ is less than $\varepsilon$, such a point $x_0$ shall be denoted by $\zeta$ and designated as a $\zeta$-point.

THEOREM II.—*The $\zeta$-points form a set of points that is* AT LEAST *dense throughout $L$.*

If this were not the case there would be an interval $(\alpha, \beta)$ in $L$ in which no $\zeta$-point appears. Let $\eta_1, \eta_2, \ldots.$ be a set of constantly decreasing positive numbers with $\lim_{i=\infty} \eta_i = 0$. Begin with $\eta_1$ as the $A$ and $(\alpha, \beta)$ as the interval of the Fundamental Theorem. Then there will be a sub-interval $(\alpha_1, \beta_1)$ of $(\alpha, \beta)$ no one of whose points is a $\gamma$-point and it follows from reasoning similar to that employed in §5 that

$$|S_n(x)| < \eta_1, \qquad \alpha_1 \leq x \leq \beta_1$$

for all values of $n$ from a fixed integer $m_1$ on. Hence the coefficient of convergence for this interval, $\bar{B}_1$, is not greater than $\eta_1$.

Next repeat the above step, taking as the $A$ of the Fundamental Theorem $\eta_2$ and as the interval $(\alpha_1, \beta_1)$. Then an interval $(\alpha_2, \beta_2)$ lying within $(\alpha_1, \beta_1)$, and an integer $m_2 \geq m_1$ will be found such that

$$|S_n(x)| < \eta_2, \qquad \alpha_2 \leq x \leq \beta_2, \qquad n \geq m_2.$$

Successive repetitions of this step lead to a series of intervals $(\alpha_i, \beta_i)$, where $\alpha_i < \beta_i$, $\alpha < \alpha_1 < \alpha_2, \ldots.$, $\beta > \beta_1 > \beta_2, \ldots.$; and a series of integers $m_1 \leq m_2 \leq m_3 \ldots.$ such that

$$|S_n(x)| < \eta_i, \qquad \alpha_i \leq x \leq \beta_i, \qquad n > m_i.$$

The $\alpha$'s and $\beta$'s may be so chosen that

$$\lim_{i=\infty} \alpha_i = \lim_{i=\infty} \beta_i = \bar{c}.$$

Then the point $\bar{c}$ is a $\zeta$-point and the interval $(\alpha, \beta)$ is not free from $\zeta$-points. From the contradiction herein contained follows the truth of the theorem.

7. Uniform and non-uniform convergence are conceptions that relate to the behavior of the variable function $s_n(x)$ *throughout an interval.* The theorems just established have brought out the importance of the role that certain *points* of the interval play with regard to the behavior of the peaks in their vicinity. Let $x_0$ be any point of $L$, $\bar{B}_h$ the coefficient of convergence of the interval $(x_0, x_0 + h)$. $\bar{B}_h$ never increases when $|h|$ decreases, and it is always positive. Hence it converges toward a limit that is positive or zero. Let

$$\lim_{h=0} \bar{B}_h = \bar{b}^+, \quad \bar{b}^-,$$

according as $h$ is positive or negative.

*Indices of a Point.* $\bar{b}^+$, $\bar{b}^-$ shall be defined as the *forward* resp. *backward index* of the point $x_0$.

If the convergence is uniform throughout an interval, $\bar{b}^+ = \bar{b}^- = 0$ at all points of the interval, and conversely, if $\bar{b}^+ = \bar{b}^- = 0$ for each point of an interval, the convergence is uniform throughout the interval.*

*This theorem is virtually contained in the opening paragraph of du Bois' paper above referred to; of course in different form, since du Bois did not have the indices of a point.

With this definition Theorems I and II may be restated as follows:

THEOREM I.—*The points of L one of whose indices is infinite form* AT MOST *a set satisfying Conditions* $(P)$.

THEOREM II.—*The points of L in which each index is* 0 *form* AT LEAST *a set dense throughout* $L$.

8. We have thus been led to two theorems that state necessary conditions for the set of $X$-points and the set of $\zeta$-points. Are these conditions, conversely, sufficient? i. e.

*a*) Given any set $G$ of points satisfying Conditions $(P)$, does there exist a function $s_n(x)$ satisfying Conditions $(A)$ and having the points $G$ (and no others) as its $X$-points?

*b*) Given any set $H$ of points dense throughout $L$, does there exist a function $s_n(x)$ satisfying Conditions $(A)$ and having the points $H$ (and no others) as its $\zeta$-points?

The first of these questions is answered in what follows in the affirmative, the second in the negative, and the complete condition (necessary and sufficient) for the $\zeta$-points is formulated in terms of a certain class of sets of points.

Before proceeding to the proofs just promised, I will give an example useful both here and in the second part of this paper.

Let a set of points $\Gamma$ be constructed as follows.* The interval $L$ shall be the interval (0, 1) (Fig. 9).

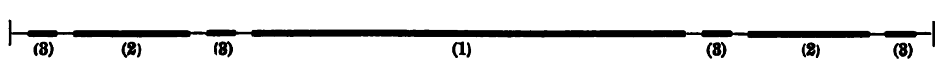


FIG. 9.

*First Step.* In the middle of this interval lay off an interval (1) of length

$$l_1 = \lambda - \tfrac{1}{3}\lambda,$$

where $\lambda$ is chosen arbitrarily as a positive quantity not greater than unity: $0 < \lambda \leqq 1$. For present purposes $\lambda$ shall be taken as $\frac{3}{4}$, $l_1 = \frac{1}{2}$.

*Second Step.* In the middle of each of the free end intervals lay off an inter-

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* This is a special case of a set given by Harnack, Math. Ann., vol. 19, p. 239.

val (2), both of these intervals to be of the same length $l_2$ and such that the total length of the intervals (1), (2) is

$$l_1 + 2l_2 = \lambda - \tfrac{1}{4}\lambda.$$

*n-th Step.* In the middle of each of the equal free intervals lay off an interval ($n$), all of these intervals to be of the same length $l_n$ and such that the total length of the intervals (1), (2), .... ($n$) is

$$l_1 + 2l_2 + 2^2 l_3 + \ldots\ldots + 2^{n-1} l_n = \lambda - \frac{\lambda}{n+2}.$$

When $n$ increases indefinitely, a set of intervals is obtained and their extremities form a set of points. If to these points those points of $L$ not already included among these extremities, but which still are limiting points of the extremities, be added and this complete set of points be denoted by $\Gamma$, then $\Gamma$ is an example of the most general set of points satisfying Conditions ($P$);—the most general, at least, in regard to the properties of such sets that are of importance here.*

A function $s_n(x)$ satisfying Conditions ($A$) and having the points of $\Gamma$ as its $X$-points shall now be constructed. To begin with, consider the function

$$\psi_n(x) = nxe^{-nx^2}, \qquad x \geqq 0.$$

The approximation curves are of the same character as those shown in Fig. 1. Next form the function†

$$\begin{aligned} \phi_n(x, l) &= \frac{\pi}{l} \sin \frac{\pi x}{l} \cdot \psi_n\left(\cos \frac{\pi x}{l}\right) & 0 \leqq x \leqq \frac{l}{2} \\ &= -\frac{\pi}{l} \sin \frac{\pi x}{l} \cdot \psi_n\left(\cos \frac{\pi x}{l}\right) & -\frac{l}{2} \leqq x \leqq 0 \\ &= 0 \text{ for all other values of } x. \end{aligned}$$

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* The objection might be raised to this example that $\Gamma$ contains no isolated points, each point of $\Gamma$ being a limiting point. It would be easy to meet this objection either by taking a less special case of Harnack's set or by adding to $\Gamma$ the points ($\lambda$ being taken $= \frac{3}{4}$):

$$\tfrac{1}{4} + \frac{1}{9+n}, \quad \tfrac{1}{2} - \frac{1}{9+n}, \quad \tfrac{1}{2}; \qquad n = 1, 2, \ldots.$$

But the example would thereby be rendered only more complicated without fulfilling any better its mission of illustration for the purposes of this paper.

† The function now to be formed is chosen partly with reference to future use. It would suffice for present purposes to omit the factors $\frac{\pi}{l} \sin \frac{\pi x}{l}$, $-\frac{\pi}{l} \sin \frac{\pi x}{l}$.

The approximation curves for this function are indicated in Fig. 10.

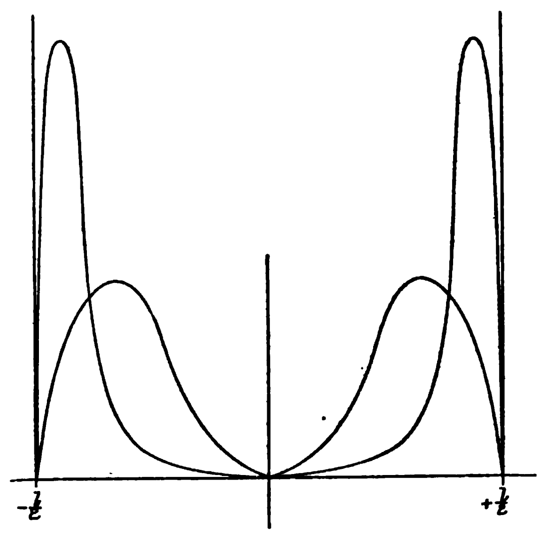


FIG. 10.

Let the middle points of the above intervals $(n)$ be denoted by

$$a_1^{(n)},\ a_2^{(n)},\ \ldots\ a_{2^n-1}^{(n)}.$$

We are now ready to form our $s_n(x)$, which we define as follows:

$$\begin{aligned} s_n(x) = \quad & \phi_n(x - a_1^{(1)}, l_1) \\ & + \phi_n(x - a_1^{(2)}, l_2) + \phi_n(x - a_2^{(2)}, l_2) \\ & + \phi_n(x - a_1^{(3)}, l_3) + \ldots\ldots\ldots + \phi_n(x - a_4^{(3)}, l_3) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \phi_n(x - a_1^{(n)}, l_n) + \ldots\ldots\ldots\ldots\ldots\ldots + \phi_n(x - a_{2^n-1}^{(n)}, l_n). \end{aligned}$$

$s_n(x)$ is continuous in $L$ and converges toward 0 for every value of $x$; for if $x_0$ is a point of any interval $(i)$, at most one term in the above sum of terms is different from 0, and this term converges toward 0 for all values of $x$. And if $x_0$ does not lie in any interval $(i)$, all the terms of this sum are 0. Every extremity of an interval is a $\mathfrak{X}$-point and hence every point of $\Gamma$ and no other point of $L$ is a $\mathfrak{X}$-point.

*ad a*). By the aid of this example it is now readily seen that if any set of points $G$ satisfying Conditions $(P)$ be given, a function $s_n(x)$ can be constructed satisfying Conditions $(A)$ and having these points as its $\mathfrak{X}$-points. For let

$\eta_1, \eta_2, \ldots.$ be a set of constantly decreasing positive quantities with $\lim_{i=\infty} \eta_i = 0$. Consider the intervals of $L$ which, with the exception of their extremities, are free from points of $G$. There will be but a finite number of such intervals whose length is greater than $\eta_1$. Denote these, taken in any order, as $(1)_1, (1)_2, \ldots. (1)_{k_1}$ (Fig. 11). Next proceed to those not already considered


FIG. 11.

whose length is greater than $\eta_2$ and denote them, taken in any order, by $(2)_1, (2)_2, \ldots. (2)_{k_2}$. And so on. Let the middle-point and the length of $(n)_m$ be denoted respectively by $a_{n,m}, l_{n,m}$. Then

$$s_n(x) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \phi_n(x - a_{i,j}, \ l_{i,j})$$

will satisfy Conditions ($A$) and have the points of $G$ as its $X$-points.

9. Before beginning the direct study of question $b$) I will make a digression, the object of which is the ascertainment of certain properties of the set of points of $L$ complementary to the $\zeta$-points.

The $X$-points were obtained (§4) by allowing $A$ to *increase*, thus sifting out from the $\gamma$-points certain ones; and those that always stayed in the sieve were the $X$-points. Here just the reverse process shall demand our attention: $A$ shall *converge toward* 0 and the set of points toward which the corresponding $\gamma$-points converge (in a sense presently to be defined accurately) shall be considered.

Let $\eta_1, \eta_2, \ldots.$ be a set of constantly decreasing positive quantities with $\lim_{i=\infty} \eta_i = 0$. Take $\eta_i$ as the $A$ of the Fundamental Theorem and denote the corresponding set of $\gamma$-points by $G_i$. Then all the points of $G_i$ are also points of $G_{i'}$, if $i' > i$; and it is natural to denote that set of points of $L$, each of which appears in some $G_i$ as $\lim_{i=\infty} G_i$. This set I will call $q$ and write

$$q = \lim_{i=\infty} G_i.$$

Thus $q$ may consist of a single point, as in Ex. 1, §2; or it may be dense throughout $L$, as in Ex. 3 of the same paragraph. $q$ contains all $X$-points.

The points of $q$ are those points of $L$ that are not $\zeta$-points; in other words, they are those points whose indices are not both $0$.

*Definition of $\xi$-Points.* A point of $q$ shall be denoted by $\xi$ and designated as a $\xi$-point.

$q$ belongs to a class of sets of points of so much importance, as will presently appear (Theorem III), that I will define this class independently of the considerations that have led to it.

*The Class of Sets of Points $Q$.* A set of points such that a variable set of points $G_i$ exists satisfying the following conditions:

1) $G_i$ satisfies Conditions $(P)$;

2) $G_{i'}$ contains *at least* all the points of $G_i$, if $i' > i$;

3) Any point of the given set appears in $G_i$, if $i$ be chosen sufficiently large—

shall be defined as belonging to the Class of Sets of Points $Q$ and denoted as a set of points $Q$.

$$Q = \lim_{i=\infty} G_i.$$

Every set of points satisfying Conditions $(P)$ is obviously a set $Q$.

10. Thus it appears that a necessary condition which the $\xi$-points satisfy is that they form a set of points $Q$. But the converse is also true, and both of these theorems shall be combined in

THEOREM III.—*The $\xi$-points form a set of points $Q$; and conversely, to every set of points $Q$ there corresponds a function $s_n(x)$ satisfying Conditions $(A)$ and having $Q$ as its $\xi$-points.*

The proof of the second part of the theorem is as follows. Let $G_\rho$ be any component set of $Q$. Then a function $\sigma_n^{(\rho)}(x)$ satisfying Conditions $(A)$ and having the points of $G_\rho$ as its $\gamma$-points can be constructed in a manner similar to that in which the function $s_n(x)$ of §8 was built up. Out of these elementary functions $\sigma_n^{(\rho)}(x)$ the desired function $s_n(x)$ is forged.

Let
$$\Psi_n(x) = \sqrt{2e}\, nxe^{-n^2x^2}, \qquad x \geq 0.$$

The altitude of the peaks is unity. Let

$$\bar{\Phi}_n(x, l) = \Psi_n\left(\cos\frac{\pi x}{l}\right), \qquad -\frac{l}{2} \leq x \leq \frac{l}{2};$$
$$= 0 \text{ for all other values of } x.$$

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Further, let

$$\sigma_n^{(\rho)}(x)=\sum_{i=1}^{n}\sum_{j=1}^{k_i^{(\rho)}}\overline{\Phi}_n(x-a_{i,j}^{(\rho)},\ l_{i,j}^{(\rho)}),$$

where $k_i^{(\rho)}$, $a_{i,j}^{(\rho)}$, $l_{i,j}^{(\rho)}$ stand here in the same relation to $G_\rho$ as $k_i$, $a_{i,j}$, $l_{i,j}$ do in §8 to $G$.

If $s_n(x)$ be now defined by the equation

$$s_n(x)=\sum_{\rho=1}^{n}\frac{1}{\rho\,!}\,\sigma_n^{(\rho)}(x),$$

then $s_n(x)$ satisfies Conditions ($A$) and has $Q$ for its $\xi$-points and no others.

That the points $Q$ are $\xi$-points is seen by the same reasoning as that employed in Ex. 3, §2. But these are the only $\xi$-points. For let $x_0$ be any other point and choose the positive quantity $\varepsilon$ at pleasure. Then

1) $\bar{\rho}$ can be so taken that

$$\frac{1}{(\bar{\rho}+1)\,!}+\frac{1}{(\bar{\rho}+2)\,!}+\ldots.<\frac{1}{\bar{\rho}\,!\,.\,\bar{\rho}}<\tfrac{1}{2}\varepsilon,$$

and hence

$$\sum_{\rho=\bar{\rho}+1}^{\infty}\frac{1}{\rho\,!}\,\sigma_n^{(\rho)}(x)<\tfrac{1}{2}\varepsilon,$$

whatever $x$ may be.

2) $\bar{\rho}$ being thus fixed, a positive quantity $\delta$ can be so determined that all points of $G_{\bar{\rho}}$ (and hence of $G_\rho$, if $\rho<\bar{\rho}$) are external to the interval $(x_0-\delta,\ x_0+\delta)$, and hence $m$ can be so taken that for all points $x$ of this interval

$$\sum_{\rho=1}^{\bar{\rho}}\frac{1}{\rho\,!}\,\sigma_n^{(\rho)}(x)<\tfrac{1}{2}\varepsilon,\qquad n>m.$$

From the addition of these last two inequalities it follows that

$$s_n(x)<\varepsilon,\qquad x_0-\delta\leqq x\leqq x_0+\delta.$$

But $s_n(x)$ is never negative. Thus $x_0$ is a $\zeta$-point and the theorem is proved.

11. *ad b*) Since the $\zeta$-points and the $\xi$-points are complementary to each other, the answer to question *b*) is implied in Theorem III and may be expressed as a corollary to that theorem.

Corollary.—*The $\zeta$-points form a set of points whose complementary set is a set $Q$; and conversely, any set of points whose complementary set is a set $Q$ is a possible set of $\zeta$-points.*

Thus the problem of determining whether a given set of points can serve as a set of $\xi$- resp. $\zeta$-points has been shown to be identical with the problem of determining whether the given set resp. its complementary set is a set belonging to the class $Q$; and so the solution of question $b$) has been made to depend on the solution of a problem in the Theory of Sets of Points.

12. Before leaving this subject I will deduce a further necessary condition for the $\zeta$-points.

Theorem IV.—*The $\zeta$-points are non-enumerable in every subinterval of $L$ whatsoever.*

Since $L$ itself was an arbitrary interval, it is sufficient to show that the $\zeta$-points are not enumerable in $L$. Suppose they were enumerable. Let them be denoted by $\zeta_1, \zeta_2, \ldots$. Add to the points $G_1$ the point $\zeta_1$; to $G_2, \zeta_1, \zeta_2$; to $G_i, \zeta_1, \zeta_2, \ldots \zeta_i$; and denote the new sets of points respectively by $G'_1, G'_2, \ldots G'_i$. Then $\lim_{i=\infty} G'_i$ is the *totality* of the points of $L$, and hence, since $G'_i$ satisfies the conditions of a component, this totality of points appears as a set $Q$. But the complementary set is nil, so that that function $s_n(x)$ which by Theorem III has $Q$ as its $\xi$-points, has no $\zeta$-points, and this is absurd.

Thus it appears that not every set of points dense throughout $L$ can serve as a set of $\zeta$-points; e. g. the rational numbers could not form a set of $\zeta$-points. But the $\xi$-points, as is easily shown by an example, may form a set non-enumerable in every interval.

In the proof of Theorem IV we have obtained incidentally a theorem regarding sets $Q$, namely: *A set $Q$ forms in no interval whatsoever a continuum. The complementary set is in every interval non-enumerable.*

Theorems II and IV express two independent necessary conditions for the $\zeta$-points. Whether these conditions are also sufficient is a question demanding further study.

## II.—The Integration of Series Term by Term.

13. The question that forms the subject of the second part of this paper is the determination of the conditions under which the $u$-series of §1 can be inte-

grated term by term; i. e. that

$$\int_{x_0}^{x_1} \lim_{n=\infty} s_n(x)\,dx = \lim_{n=\infty} \int_{x_0}^{x_1} s_n(x)\,dx,$$

when $$a \leq x_0 < x_1 \leq b;$$

or expressed in terms of $S_n(x)$, that

$$\int_{x_0}^{x_1} \lim_{n=\infty} S_n(x)\,dx = \lim_{n=\infty} \int_{x_0}^{x_1} S_n(x)\,dx.$$

Geometrically the left-hand side of these equations signifies the area under the curve $y = f(x)$ resp. $y = 0$ between $x = x_0$ and $x = x_1$; the right-hand side, on the other hand, is the *limit* that the corresponding area under the *variable* curve $y = s_n(x)$ resp. $y = S_n(x)$ approaches when $n = \infty$.

Since $\lim\limits_{n=\infty} S_n(x) = 0$, $\int_{x_0}^{x_1} \lim\limits_{n=\infty} S_n(x)\,dx = 0$ and the question of Part II in its reduced form is: *Under what conditions will*

$$\lim_{n=\infty} \int_{x_0}^{x_1} S_n(x)\,dx = 0?$$

14. Let us begin with some simple examples.

Ex. 1. $$S_n(x) = nxe^{-nx^2},$$

$$\int_0^x S_n(x)\,dx = \tfrac{1}{2}(1 - e^{-nx^2}),$$

$$\lim_{n=\infty} \int_0^x S_n(x)\,dx = \tfrac{1}{2} \neq 0.$$

Thus although the integrand converges toward 0 for all values of $x$, the integral does not, if $x = 0$ forms one of the limits of the integration. Geometrically the area under the curve $y = nxe^{-nx^2}$ in the interval $(0, x)$ converges toward $\frac{1}{2}$.

If $$S_n(x) = n^2xe^{-nx^2},$$

$$\lim_{n=\infty} \int_0^x S_n(x)\,dx = \infty.$$

If however $$S_n(x) = \sqrt{n}\,xe^{-nx^2},$$

$$\lim_{n=\infty} \int_0^x S_n(x)\,dx = 0,$$

and the same is true if $S_n(x) = \dfrac{n^2x}{1 + n^3x}$.

In the first, second and fourth cases the peaks are infinite (Fig. 1), in the third they are not (Fig. 2).

Ex. 2. $$S_n(x) = \phi_n(x, l), \qquad (\S 8)$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} S_n(x)\,dx = 2\int_0^{\frac{l}{2}} \frac{\pi}{l}\, n \sin\frac{\pi x}{l} \cos\frac{\pi x}{l}\, e^{-n\cos^2\frac{\pi x}{l}}\,dx = \int_{-n}^{0} e^{\xi} d\xi = 1 - e^{-n},$$

where
$$\xi = -n\cos^2\frac{\pi x}{l};$$

hence
$$\lim_{n=\infty}\int_{-\frac{l}{2}}^{\frac{l}{2}} S_n(x)\,dx = 1.$$

It thus appears that when there is a single point in $L$ in whose neighborhood peaks occur,

$$\int_{x_0}^{x_1} S_n(x)\,dx$$

may, if the peaks become infinite, converge toward $0$, toward a limit different from $0$, or diverge. If the peaks remain finite, then it is at once evident geometrically that the limit must be $0$.

15. Turning now to the general case, we recall that the $X$-points of $L$ form at most a set nowhere dense in $L$ (Theorem I); and secondly, if an interval $(\alpha, \beta)$ is free from such points, the approximation curves $y = S_n(x)$ will all lie within a finite belt bounded by $y = \pm B$ (§5). If the limits of integration lie within such an interval $(\alpha, \beta)$, then we might be inclined, generalizing from the corresponding example above, to regard it as extremely plausible that in this case $\lim\limits_{n=\infty}\int_{x_0}^{x_1} S_n(x)\,dx = 0$. But a more careful study, in the light of Part I of this paper, of the possible manner of the convergence of $S_n(x)$ toward its limit,* diminishes materially the plausibility of this inference, and in fact the proof here given *that in this case* $\lim\limits_{n=\infty}\int_{x_0}^{x_1} S_n(x)\,dx$ *actually is* $0$, although in its beginnings

* It is of great importance at this point to picture to oneself the successive approximation curves $y = S_n(x)$ of §8. An appreciation of the possibilities thereby brought to light would have saved du Bois from the errors he made in the paper above referred to (§4, foot-note).

suggested by intuition, in its further course makes clear that we have here to do with relations that transcend the bounds of our present intuition.

16. The plan of the proof is as follows. Let the positive quantity $\varepsilon$ be chosen at pleasure; then there exists a corresponding fixed integer $m$ such that

$$-\varepsilon < \int_{x_0}^{x_1} S_n(x)\,dx < \varepsilon, \qquad n > m.$$

For if this were not the case, one of the inequalities, say the right-hand one, would be violated for an infinite sequence of values of $n$: $n_1, n_2, \ldots$. Choose the interval of the Fundamental Theorem as $(x_0, x_1)$ and take $A < \frac{\varepsilon}{x_1 - x_0}$. Then those portions of the approximation curve

$$y = S_{n_i}(x), \qquad i = 1, 2, \ldots$$

which lie above the line $y = A$ would, together with the corresponding segments of this line bound an area (the shaded area of Fig. 12) which could not converge toward 0 when $i$ increases. For

$$\int_{x_0}^{x_1} S_{n_i}(x)\,dx \geqq \varepsilon > A.(x_1 - x_0)$$

and $A.(x_1 - x_0)$ is the area below the line $y = A$. I show however that this latter inequality is impossible (i. e. that the shaded area of Fig. 12 *must* converge

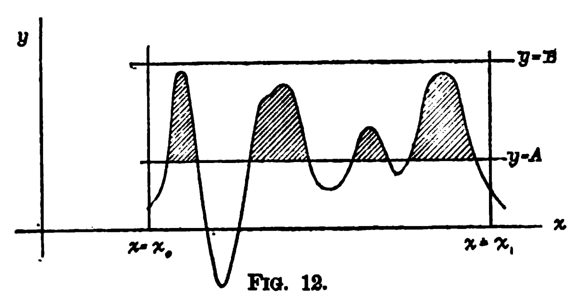

FIG. 12.

toward 0). Hence the above supposition is untenable and from this the truth of the right-hand inequality follows. The same reasoning applies to the left-hand inequality. But this double-inequality is only another form of statement of the theorem itself.

17. A preliminary study of some further properties of the most general set of points $G$ satisfying Conditions $(P)$ is necessary.

We have already enumerated the intervals $(n)_m$ of $L$ (§8) whose extremities are points of $G$, but which contain in their interior no further points of $G$. Let

$$l_{n,1}+l_{n,2}+\ldots+l_{n,k_n}=\lambda_n,$$

$$\sum_{n=1}^{\infty}\lambda_n=\lambda.$$

$\lambda$ is surely not greater than $l$, but, as in the example of §8, it may be less. The points of $G$ may be enclosed in a finite number of intervals, the sum of whose lengths exceeds $l-\lambda$ by less than the arbitrary quantity $\delta$. For let $p$ be so taken that

$$\sum_{i=p+1}^{\infty}\lambda_i<\tfrac{1}{2}\delta.$$

Then

$$\sum_{i=1}^{p}\lambda_i>\lambda-\tfrac{1}{2}\delta.$$

Out of each of the $\mu$ intervals whose sum is $\sum_{i=1}^{p}\lambda_i$ a subinterval $\theta_\rho$ may be selected, neither of whose extremities coincides with those of the interval in which it lies and of such length that

$$\sum_{\rho=1}^{\mu}\theta_\rho>\sum_{i=1}^{p}\lambda_i-\tfrac{1}{2}\delta>\lambda-\delta.$$

The interval $L$ is thus divided into $\mu$ subintervals $\theta_\rho$, the sum of whose lengths

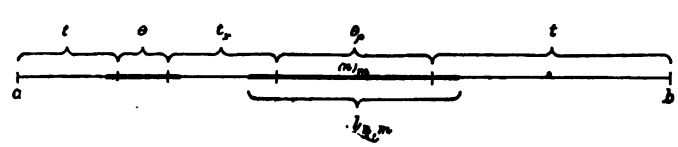


FIG. 18.

lies between $\lambda$ and $\lambda-\delta$ and which, inclusive of their extremities, contain no point of $G$. Thus the points of $G$ fall in the remaining intervals, $t_r$, let us call them, finite in number, whose sum $\sum t_r$ lies between $l-\lambda$ and $l-\lambda+\delta$. And if any set of intervals, finite in number, be so taken as to enclose in their interiors (not on their boundaries) all the points of $G$, then their sum must be greater than $l-\lambda$; for the above set of $\theta$-intervals can always be so taken that all points of the corresponding $t$-intervals lie within this set.

On the ground of these relations, $l-\lambda$ is defined as the *content** of $G$. Let it be denoted by $I$:

$$I=l-\lambda.$$

18. LEMMA. Let $G$ be any set of points satisfying Conditions $(P)$, $G_i$ a component of $G$:

$$\lim_{i=\infty} G_i = G;$$

and let the content of $G$, $G_i$ be denoted respectively by $I$, $I_i$. Then

$$\lim_{i=\infty} I_i = I.$$

This theorem is far from being self-evident. Suppose we had, instead of $G$, the set of rational numbers $R$ in the interval $(0, 1)$ and developed this set into the series

$$R=R_1+R_2+\ldots\ldots,$$

where $R_1=(\frac{1}{2})$, $R_2=(\frac{1}{3},\frac{2}{3})$, $R_3=(\frac{1}{4},\frac{3}{4})$, $R_4=(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$, etc.,

$$G_i=R_1+R_2+\ldots\ldots+R_i.$$

Then $\qquad I_i=0$; but $I=1$.

The lemma is proved as follows: Since $I_i \leqq I$ and $I_{i'} \geqq I_i$, if $i'>i$, $I_i$ approaches a limit when $i$ becomes infinite. Let $\lim\limits_{i=\infty} I_i = I'$. Then $I' \leqq I$.

Let the positive quantity $\bar{\delta}$ be chosen at pleasure and then let the constantly decreasing quantities $\bar{\delta}_1, \bar{\delta}_2, \ldots\ldots$ be so taken that

$$\bar{\delta}_1+\bar{\delta}_2+\ldots\ldots=\bar{\delta}.$$

The points of $G_1$ can be enclosed in a finite number of intervals $\tau_\kappa^{(1)}$ whose sum $\sum\limits_\kappa \tau_\kappa^{(1)}$ is less than $I_1+\bar{\delta}_1$ and whose extremities are not points of $G$; the neighborhood of each extremity will then also be free from the points of $G$. The points of $G_2$ not already lying in the intervals $\tau_\kappa^{(1)}$ can be enclosed in a finite

* Harnack, Math. Ann., vol. 25; Cantor, ibid. vol. 23, p. 473.

number of further intervals $\tau_\kappa^{(2)}$ whose extremities are not points of $G$ and such that

$$\sum_\kappa \tau_\kappa^{(1)} + \sum_\kappa \tau_\kappa^{(2)} < I_2 + \overline{\delta}_1 + \overline{\delta}_2.$$

For it becomes evident on a little reflection that if $I_2'$, $I_2''$ denote respectively the content of that portion of $G_2$ lying in the intervals $\tau_\kappa^{(1)}$, $\tau_\kappa^{(2)}$, then $I_2' + I_2'' = I_2$. And $I_2' \geqq I_1$. It is sufficient therefore to make $\sum_\kappa \tau_\kappa^{(2)} < I_2'' + \overline{\delta}$.

The repetition of this step leads to a set of intervals $\tau$, finite in number, which include in their interior all the points of $G_i$ and whose extremities, not being themselves points of $G$, lie each within an interval free from points of $G$; and furthermore, the intervals $\tau_\kappa^{(j)}$ remain unchanged fcr all values of $i > j$, new intervals $\tau_\kappa^{(i')}$, $i' > j$, being merely added. Finally

$$\sum_\kappa \tau_\kappa^{(1)} + \sum_\kappa \tau_\kappa^{(2)} + \ldots + \sum_\kappa \tau_\kappa^{(i)} < I_i + \overline{\delta}_1 + \overline{\delta}_2 + \ldots + \overline{\delta}_i < I' + \overline{\delta}.$$

As $i$ increases indefinitely, the number of the $\tau$-intervals does not become infinite, but reaches a certain maximum number $M$. For if the number were to become infinite, the extremities of these intervals would have to cluster about at least one limiting point $x'$, and it would be possible to choose out of the intervals that thus cluster about $x'$ a set of points of $G$ with $x'$ as their limiting point. Hence $x'$ would itself be a point of $G$. But $x'$ lies in no $\tau$-interval, and thus a contradiction results from the supposition that the number of the $\tau$-intervals is infinite.

The $M$ $\tau$-intervals contain in their interior (i. e. exclusive of their extremities) all the points of $G$. For if, as $i$ increases, the points of $G_i$ in a certain interval were to converge toward the extremity of the interval, then this point would also be a point of $G$. Thus all the points of $G$ are enclosed within $m$ intervals, the sum of whose lengths is less than $I' + \overline{\delta}$, a quantity which, if $I' < I$, can by proper choice of $\overline{\delta}$ be made less than $I$. But this is impossible (§17). Hence $I' = I$.

The set of points $G$ to which the lemma is to be applied is the set of $\gamma$-points of §3, the component $G_i$ being obtained as follows. Let $\gamma$ denote an arbitrary point of $G$ and let $j$ be the smallest integer for which

$$|S_n(\gamma)| \leqq A, \qquad n \geqq j.$$

Then those $\gamma$-points corresponding to values of $j \leqq i$ form a set $G_i$ satisfying Conditions ($P$). For, being points of $G$, they are nowhere dense and evidently any point $\bar{\gamma}$ toward which such a set of $\gamma_i$-points converges is itself a $\gamma_i$-point.

19. We are now ready to enter on the immediate proof that the inequality

$$\int_{x_0}^{x_1} S_{n_i}(x)\,dx \geqq \varepsilon > A.(x_1 - x_0), \qquad i = 1, 2, \ldots.$$

is impossible. Let $\varepsilon_1$ be so taken that

$$\varepsilon_1 < \varepsilon - A.(x_1 - x_0).$$

It is sufficient to show that the shaded area lying above the line $y = A$ of Fig. 12, which shall be denoted by $C_n$, becomes and remains less than $\varepsilon_1$ as $n$ increases, since whatever area lies below this line is surely less than $A(x_1 - x_0)$.

First choose $\delta$ at pleasure, construct the corresponding $\theta$- and $t$-intervals, and then take $m$ so that

$$S_n(x) \leqq A, \qquad n \geqq m$$

for all points $x$ lying in the $\theta_p$-intervals. In the most unfavorable class of cases (illustrated in Fig. 14$a$. In Figs. 14$a$, 14$b$ the $\gamma$-points are plotted on the line

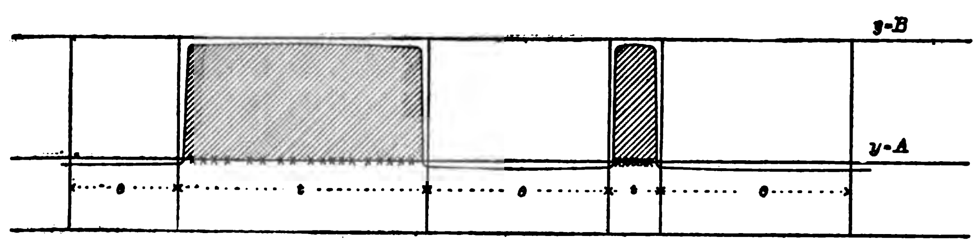


FIG. 14$a$.

$y = A$ instead of on the $x$-axis in order to indicate more clearly for what ones of these points $S_n(x)$ is greater and for what ones less than $A$.)

$$G_1 = G_2 = \ldots. = G_m = 0, \qquad I_m = 0,$$

the curve $y = S_m(x)$ spanning all the points of $G$ and rising so abruptly as to make $C_m$ come indefinitely near to the value

$$(B - A).\sum_r t_r < (B - A)(I + \delta).$$

Next let $n = m + p$. The curve $y = S_{m+p}(x)$ (Fig. 14*b*) lies above the

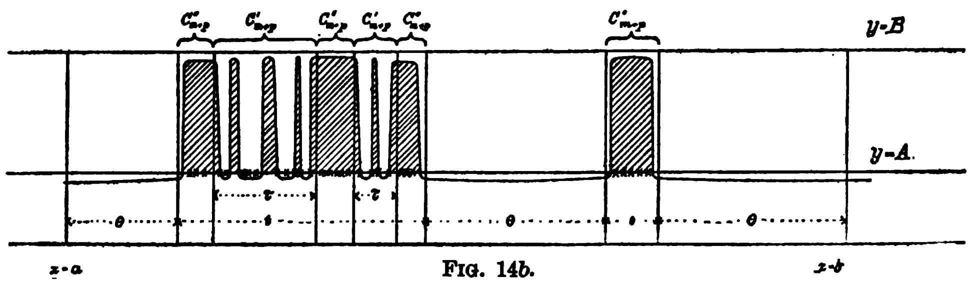


FIG. 14*b*.

line $y = A$ only in the $t$-intervals. Let the $\tau$-intervals be constructed as in §18, $\bar{\delta}$ being assumed arbitrarily, but let them be so taken as always to lie within the $t$-intervals already assumed.

$$I_{m+p} < \sum_{\kappa} \tau_{\kappa}^{(1)} + \sum_{\kappa} \tau_{\kappa}^{(2)} + \ldots + \sum_{\kappa} \tau_{\kappa}^{(m+p)} < I_{m+p} + \bar{\delta}.$$

Erect perpendiculars to the $x$-axis at the extremities of the $\tau$-intervals. $C_{m+p}$ is thereby divided into two parts: $C'_{m+p}$ lying above the $\tau$-intervals and $C''_{m+p}$ lying above the complementary intervals.

First consider any $\tau$-interval. The curve $y = S_{m+p}(x)$ can rise above the line $y = A$ only in such portions of the interval as are free from points of $G_{m+p}$. If such intervals be added together for all the intervals $\tau$, then the *limit* of their sum is but

$$\sum_{\kappa} \tau_{\kappa}^{(1)} + \sum_{\kappa} \tau_{\kappa}^{(2)} + \ldots + \sum_{\kappa} \tau_{\kappa}^{(m+p)} - I_{m+p} < \bar{\delta},$$

this inequality holding for all values of $p$. The area under a continuous curve is the limit approached by the sum of the inscribed rectangles. The portions of the rectangles inscribed in $y = S_{m+p}(x)$ that lie above the line $y = A$ have an area less than

$$(B - A)\Big(\sum_{\kappa} \tau_{\kappa}^{(1)} + \ldots + \sum_{\kappa} \tau_{\kappa}^{(m+p)} - I_{m+p}\Big).$$

Hence their limit $C'_{m+p}$ cannot exceed this quantity, and so

$$C'_{m+p} < (B - A)\bar{\delta}.$$

On the other hand, the sum of the lengths of the complementary intervals is

$$\sum_{r} t_r - \Big(\sum_{\kappa} \tau_{\kappa}^{(1)} + \ldots + \sum_{\kappa} \tau_{\kappa}^{(m+p)}\Big) < I + \delta - I_{m+p},$$

and hence $$C''_{m+p} < (B - A).(I + \delta - I_{m+p}).$$

Therefore since $C'_{m+p} + C''_{m+p} = C_{m+p}$,

$$C_{m+p} < (B - A).(I - I_{m+p} + \delta + \bar{\delta}).$$

Let the positive quantity $\eta$ be taken at pleasure. Then by the lemma $p_1$ can be so taken that

$$I - I_{m+p} < \eta, \qquad p > p_1.$$

Hence if

$$\eta + \delta + \bar{\delta} < \frac{\varepsilon_1}{B - A},$$

$$C_n < \varepsilon_1, \qquad n > m + p_1.$$

In a similar manner it is shown that the first inequality of §16 is impossible.

Thus the theorem of §15 has been established. It may be stated as follows:

THEOREM V.—*If $s_n(x)$ is any function of $x$ satisfying Conditions (A), $(\alpha, \beta)$ any interval free from X-points, and $x_0 < x_1$ any two points of this interval, then*

$$\int_{x_0}^{x_1} \lim_{n=\infty} s_n(x) . dx = \lim_{n=\infty} \int_{x_0}^{x_1} s_n(x)\, dx.$$

*If $s_n(x)$ is given in the form of a series*

$$s_n(x) = u_1(x) + u_2(x) + \ldots . + u_n(x),$$

*the series is said to be integrable term by term.*

But more than this: *It follows from the method above set forth for the proof of this theorem that, if $x_0$, $x_1$ are regarded as variable,*

$$\int_{x_0}^{x_1} s_n(x)\, dx$$

*converges* UNIFORMLY *toward its limit when n becomes infinite.*

For let $S_n(x) = S_n^+(x) + S_n^-(x)$, where $S_n^+(x) = S_n(x)$ when $S_n(x) \geqq 0$ and vanishes for all other values of $x$. $S_n^-(x) \leqq 0$. If $m_1$ be so taken that

$$\int_\alpha^\beta S_n^-(x)\, dx < \varepsilon, \qquad n > m_1,$$

then

$$\int_{x_0}^{x_1} S_n(x)\, dx < \varepsilon, \qquad \alpha \leqq x_0 < x_1 \leqq \beta.$$

If $m_2$ be so taken that

$$-\varepsilon < \int_\alpha^\beta S_n^-(x)\, dx, \qquad n > m_2,$$

then

$$-\varepsilon < \int_{x_0}^{x_1} S_n(x)\, dx.$$

And if the larger of the integers $m_1$, $m_2$ be denoted by $m$, then

$$-\varepsilon < \int_{x_0}^{x_1} S_n(x)\,dx < \varepsilon, \qquad \alpha \leqq x_0 < x_1 \leqq \beta, \qquad n > m.$$

20. The question of §13 having been answered for the case that the interval $(\alpha, \beta)$ is free from $X$-points, we now proceed to the case that $X$-points are present. Here the integral may diverge as $n$ increases. If it converges, let

$$F(x, x_0) = \lim_{n=\infty} \int_{x_0}^{x} S_n(x)\,dx.$$

If $x_0$, $x$ lie in an interval $(\alpha, \beta)$ that contains only a finite number of $X$-points, and if $F(x, x_0)$ does not vanish for all values of $\alpha \leqq x \leqq \beta$, then evidently (as in the Exs. of §14) $F(x, x_0)$ will in general be continuous, but may have a finite discontinuity, as indicated in Fig. 15, at one or more of the $X$-points. But the

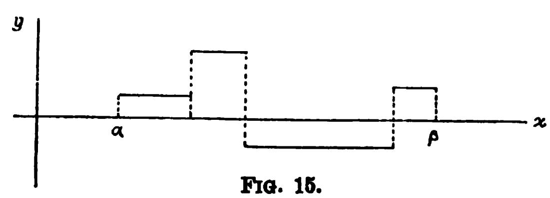


FIG. 15.

consideration of the most general set $G$ of $X$-points (§8) suggests the question: *Is it possible for $F(x, x_0)$ to be a continuous function of $x$ throughout $(\alpha, \beta)$ without vanishing for all values of $x$?*

I answer this question in the *affirmative* by giving an example of such a function. Let the set of points $\Gamma$ and the intervals $(i)$ be the same as in the example of §8, but let

$$s_n(x) = \nu\phi_n(x - a_1^{(n)}, l_n) + \dots + \nu\phi_n(x - a_{2^{n-1}}^{(n)}, l_n),$$

where $\nu = \dfrac{1}{2^{n-1}}$. This amounts to throwing the peaks all into the intervals $(n)$, all other intervals being free.

It is necessary to show 1) that $F(x, x_0)$ exists, i. e. that $\int_{x_0}^{x} s_n(x)\,dx$ converges toward a limit for all values of $x$, $x_0$ belonging to $L$; 2) that $F(x, x_0)$ is a continuous function of $x$ throughout the interval.

Since $$\int_0^x - \int_0^{x_0} = \int_{x_0}^{x},$$

it is sufficient to show that $\int_0^x s_n(x)\,dx$ converges toward a limit $F(x)$ which is a continuous function of $x$.

The symmetry with which the function $s_n(x)$ was constructed will now stand us in good stead. In fact we see at once from the expressions already obtained for the area under each of the component curves $\nu\phi_n(x - a_i^{(n)}, l_n)$ that go to make up $s_n(x)$, namely $\nu$, that

$$\int_0^1 s_n(x)\,dx = 1; \qquad \therefore\ F(1) = 1. \qquad \text{And } F(0) = 0.$$

Furthermore, from the symmetry of $y = s_n(x)$ with regard to the line $x = \frac{1}{2}$, it is evident that

$$\int_0^{\frac{1}{2}} s_n(x)\,dx = \tfrac{1}{2}\int_0^1 s_n(x)\,dx = \tfrac{1}{2}; \qquad \therefore\ F(\tfrac{1}{2}) = \tfrac{1}{2}.$$

Let $x$ be any point of the interval (1).

$$\int_0^x s_n(x)\,dx = \int_0^{\frac{1}{2}} s_n(x)\,dx + \int_{\frac{1}{2}}^x s_n(x)\,dx.$$

The second of these integrals is 0 for all values of $n$ greater than 1, and hence

$$\int_0^x s_n(x)\,dx = \tfrac{1}{2}, \qquad n > 1;$$

$$\therefore\ F(x) = \tfrac{1}{2}, \qquad a_1^{(1)} - \frac{l_1}{2} \leqq x \leqq a_1^{(1)} + \frac{l_1}{2}.$$

It is possible to generalize this result at once for any interval. Let it be the $j^{\text{th}}$ of the intervals $(i)$, and let $x$ be any point of this interval. Then

$$\int_0^x s_n(x)\,dx = \int_0^{a_j^{(i)}} s_n(x)\,dx + \int_{a_j^{(i)}}^x s_n(x)\,dx.$$

Now, as soon as $n$ is greater than $i$, $s_n(x) = 0$ throughout each interval $(i)$, and the second integral on the right vanishes.

Consider the first integral

$$\int_0^{a_j^{(i)}} s_n(x)\,dx.$$

The intervals (1), (2), (3), .... $(i)$ divide the interval $L$ up into $K_i$ complementary intervals. Let $r_{i,j}$ denote the number of these intervals to the left of $a_j^{(i)}$. As soon as $n$ exceeds the value $i$, the total area (unity) under $s_n(x)$ in the whole interval $L$ is apportioned equally to each of these complementary intervals. Hence

$$\int_0^{a_j^i} s_n(x)\,dx = \frac{r_{i,j}}{K_i}, \qquad n > i,$$

and therefore

$$\lim_{n=\infty}\int_0^{a^{(i)}} s_n(x)\,dx = \frac{r_{i,j}}{K_i}.$$

Thus

$$F(x) = \frac{r_{i,j}}{K_i}, \qquad a_j^{(i)} - \frac{l_i}{2} \leqq x \leqq a_j^{(i)} + \frac{l_i}{2}$$

and $F(x)$ exists and is continuous for all values of $x$ within any interval $(i)$, and at the extremity it is at least continuous on the side on which the interval lies. Moreover it is evident from the foregoing that if $x_0$, $x_1$ are any two values of $x$ for which $F(x)$ has already been shown to exist and if

$$x_0 < x_1, \text{ then } F(x_0) \leqq F(x_1).$$

Finally let $0 < x' < 1$ be any point not belonging to any interval $(i)$. Then in general $x'$ lies between two of the intervals $(i)$. Let the middle points of these intervals be denoted by $\alpha_i$, $\beta_i$, where $\alpha_i < \beta_i$. If however no one of the intervals $(i)$ lies to the left (right) of $x'$, let $\alpha_i = 0$ $(\beta_i = 1)$. Then

$$\int_0^{\alpha_i} s_n(x)\,dx \leqq \int_0^{x'} s_n(x)\,dx \leqq \int_0^{\beta_i} s_n(x)\,dx,$$

and this double inequality can be written, as soon as $n > i$:

$$F(\alpha_i) < \int_0^{x'} s_n(x)\,dx < F(\beta_i).$$

As $i$ increases $\alpha_i < x'$ and $\beta_i > x'$ both converge toward $x'$ as their limit; for $x'$ being a point of no interval $(i)$, the intervals $(i)$ cluster about $x'$ on both sides. Let $\alpha_i'$ denote the largest of the quantities $\alpha_1, \alpha_2, \ldots. \alpha_i$; $\beta_i'$, the smallest of the quantities $\beta_1, \beta_2, \ldots. \beta_i$. Then

$$\alpha_1' \leqq \alpha_2' \leqq \ldots. < x', \ \lim_{i=\infty} \alpha_i' = x'; \qquad \beta_1' \geqq \beta_2' \geqq \ldots. > x', \ \lim_{i=\infty} \beta_i' = x',$$

$$F(\alpha_i') < \int_0^{x'} s_n(x)\,dx < F(\beta_i'), \qquad n > i.$$

And

$$F(\alpha_1') \leqq F(\alpha_2') \leqq \ldots.; \quad F(\beta_1') \geqq F(\beta_2') \geqq \ldots.$$

Thus it appears that $F(\alpha_i')$, $F(\beta_i')$ both converge towards limits as $i$ becomes infinite; and since

$$F(\beta_i') - F(\alpha_i') \leqq F(\beta_i) - F(\alpha_i) = \frac{1}{K_i},$$

these limits are equal. Therefore $\int_0^{x'} s_n(x)\,dx$ converges toward the same limit and $F(x')$ exists.

Thus $F(x)$ exists for all values of $x$. It is easily seen that, generally, if

$$0 \leqq x_0 < x_1 \leqq 1, \quad F(x_0) \leqq F(x_1).$$

At the above point $x'$, $F(x)$ is continuous, for

$$\lim_{i=\infty} F(a_i') = F(x'),$$

and if

$$a_i' < x < x',$$

$$F(a_i') < F(x) \leqq F(x').$$

Hence

$$\lim_{x=x'} F(x) = F(x'),$$

$x$ being always less than $x'$. It is shown in a similar manner that if $x > x'$, the same equation holds.

It has been shown that if $x''$ is the extremity of one of the intervals $(i)$, $F(x)$ is continuous at $x''$ on the side on which the interval lies. It remains to show that $F(x)$ is continuous on the other side also, and that $F(x)$ is continuous for $x = 0$ and $x = 1$. The method above employed for the case of the point $x'$ is immediately applicable, the only modification being that only one half of the double inequality is necessary.

The function $F(x)$ is therefore a continuous function of $x$ which never decreases as $x$ increases and such that $F(0) = 0$, $F(1) = 1$. Thus the answer promised to the question raised at the beginning of this paragraph has been given.

It may be remarked that $F(x)$ is constant throughout each of the $\theta$-intervals and hence those points $x'$ in whose neighborhood $F(x) \neq F(x')$ can be enclosed in a finite number of $t$-intervals whose sum is less than $l - \lambda + \delta$, $\delta$ being an arbitrarily chosen positive quantity. If $\lambda$ is taken equal to $l$, this sum will be less than $\delta$, i. e. arbitrarily small.

21. We have already seen that when the $X$-points are finite in number, $F(x, x_0)$ cannot be continuous unless it vanishes for all values of $x$. Evidently the same would be true if the $X$-points were the points $\frac{1}{n}$, $0$. For, $F(x, x_0)$ being assumed to exist for all values of $x$, $x_0$ in the interval $(0, 1)$, it is easily shown that

$$F(x, 0) = F(x_0, 0) + F(x, x_0).$$

If $x_0 > 0$, $F(x, x_0) = 0$, there being but a limited number of $X$-points in the interval $(x_0, x)$. Further,

$$F(x, 0) = \lim_{x_0 = 0} F(x_0, 0) = 0$$

and hence if $F(x, x_0)$ is continuous, it vanishes for all values of $x$.

From this example we can clearly generalize as follows. $F(x, x_0)$ cannot be continuous unless it vanishes for all values of $x$, if the derivative $G^{(1)}$ of the set $G$ of $X$-points consists of a finite number of points, i. e. if the second derivative $G^{(2)} \equiv 0$.

But even if $G^{(1)}$ contains an infinite number of points, the same will be true, provided $G^{(2)}$ contains but a finite number of points, i. e. that $G^{(3)} \equiv 0$. And by induction we infer that $F(x, x_0)$ cannot be continuous unless it vanishes for all values of $x$, if the set of $X$-points is *reducible*; i. e. if $G^{(\alpha)} \equiv 0$, where $\alpha$ denotes a number of the first or second class.*

Suppose on the other hand the set $G$ of $X$-points is *perfect*.* Then I say it is always possible for $F(x, x_0)$ to be continuous without vanishing for any other value of $x$ than the value $x = x_0$. Let $G$ be included in the interval $(a, b)$, $L$, where $a, b$ are points of $G$. Select at pleasure an interval (1) whose extremities are points of $G$ not coinciding with $a, b$, but which contains no other points of $G$. Out of each of the free intervals complementary to (1) select, in the same manner as (1) was selected from $L$, intervals $(2)_1$, $(2)_2$. Repeat this process, obtaining the set $(i)_1, (i)_2, \ldots . (i)_{2^i-1}$. Then the function

$$s_n(x) = \sum_{j=1}^{2^n-1} \nu\phi_n(x - a_j^{(n)}, l_{n,j})$$

is such that $F(x, x_0)$ is continuous without vanishing for any other value of $x$ than $x_0$. The proof is similar to that given in §20. The $X$-points of $s_n(x)$ are all points of $G$, and if the intervals $(i)_k$ are so taken that each subinterval of $L$ that contains in its interior no points of $G$ appears in this list, each point of $G$ will be a $X$-point.

Now Cantor has shown* that if the first derivative $P^{(1)}$ of a set of points $P$ is enumerable, there always exists a number $\alpha$ of the first or second class such that $P^{(\alpha)} \equiv 0$; but if $P^{(1)}$ is non-enumerable, then $P^{(1)}$ contains among its points

*Cantor, Fondaments d'une théorie générales des ensembles, Acta Math., vol. 2, p. 405; Sur divers théorèmes de la théorie des ensembles, etc., ibid. p. 409 et seq.; Bendixson, Quelques théorèmes de la théorie des ensembles de points, ibid. p. 419.

a perfect set of points: $P^{(1)} = R + S$, where $R$ does not concern us and $S$ is perfect. Since $G$ contains its derivative $G^{(1)}$, $G$ will be enumerable when* and only when $G^{(1)}$ is enumerable. We are thus led to the following theorem.

THEOREM VI.—*If* $s_n(x)$ *is a function satisfying Conditions* $(A)$ *and* $\lim\limits_{n=\infty}\int_{x_0}^{x} s_n(x)\,dx = \overline{F}(x, x_0)$; *then it is a sufficient condition for the existence of the relation*:

$$\int_{x_0}^{x}\Big[\lim_{n=\infty} s_n(x)\Big]dx = \lim_{n=\infty}\Big[\int_{x_0}^{x} s_n(x)\,dx\Big]$$

*that* 1) $\overline{F}(x, x_0)$ *be a continuous function of* $x$ *and* 2) *the set* $G$ *of* $X$*-points of the interval* $(a, b)$ *be enumerable.*

*But if the set* $G$ *of* $X$*-points is non-enumerable, then there always exist functions* $s_n(x)$ *satisfying Conditions* $(A)$ *and having* $G$ *as their* $X$*-points, for which* $\overline{F}(x, x_0)$ *is continuous, but*

$$\int_{x_0}^{x}\Big[\lim_{n=\infty} s_n(x)\Big]dx \neq \lim_{n=\infty}\Big[\int_{x_0}^{x} s_n(x)\,dx\Big]$$

*if* $x \neq x_0$.

*If* $s_n(x)$ *be given as a series:*

$$s_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x),$$

*then it is a sufficient condition for the integrability of the series term by term that* 1) *the series*

$$\int_{x_0}^{x} u_1(x)\,dx + \int_{x_0}^{x} u_2(x)\,dx + \ldots$$

*be a continuous function of* $x$; 2) *the* $X$*-points be enumerable.*

22. The theorems obtained in Part II of this paper lead to corresponding theorems regarding the differentiation of a series term by term. For example:

THEOREM.—*If* $U_i(x)$ *has for all values of* $x$ *in the interval* $(a, b)$: $a \leqq x \leqq b$ *a derivative* $U_i'(x) = u_i(x)$ *continuous at each point of the interval and if the series*

$$U(x) = U_1(x) + U_2(x) + \ldots$$

*converges for all values of* $x$ *in the interval toward the limit* $U(x)$; *if further the series of the derivatives*

$$u(x) = u_1(x) + u_2(x) + \ldots$$

*Cantor, Sur les ensembles infinis et linéaires de points, ibid. p. 374.

*converges for all values of $x$ toward the continuous function $u(x)$; then in general the function $U(x)$ will have a derivative given by the differentiation of the $U$-series term by term. The points for which this fails to be true form a set $G$ satisfying Conditions $(P)$.*

*And conversely, if $U(x)$, $U_i(x)$ are functions having a continuous derivative $U'(x) = \bar{u}(x)$, $U_i'(x) = u_i(x)$ throughout $(a, b)$ and if the series of the derivatives*

$$u(x) = u_1(x) + u_2(x) + \dots.$$

*converges toward a continuous function $u(x)$, then the $U$-series can be differentiated term by term.*

For the continuous function $\bar{u}(x) - u(x)$ differs from 0 at most in a set of points $G$ satisfying Conditions $(P)$ and hence is 0 for all values of $x$.

HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS, *August* 1896.

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[NOTE.—In a second paper presented to the American Mathematical Society at its Summer Meeting the geometrical method for the study of uniform convergence here set forth was treated at some length. This paper has since been printed in the Bulletin of the Society, 2d Ser., vol. III, pp. 59–86; November 1896.]

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### I.—*Non-Uniform Convergence.*

# *Simple Proof of a Fundamental Theorem in the Theory of Functions.*

BY R. D. BOHANNAN, *Ohio State University, Columbus, O.*

"If a Riemann's Surface is reduced by $m$ cross-cuts into $n$ distinct simply connected pieces, and by $m'$ cross-cuts into $n'$ such pieces, then $m - n = m' - n'$."

($a$) In a simply connected surface, one cross-cut makes two simply connected pieces, and $n$ cross-cuts make $n + 1$ such pieces.

($b$) In a $p$-ply connected surface, $p - 1$ *appropriate* cross-cuts *are necessary* to reduce it to a simply connected surface.

($c$) If a $p$-ply connected surface has been reduced by $m$ cross-cuts to $n$ simply connected pieces, the $p - 1$ cross-cuts noted in ($b$) have been made. Thus the $n$ simply connected pieces are due to $m - (p - 1)$ of the cuts. Thus by ($a$),

$$n = m - (p - 1) + 1$$

$$\therefore \; m - n = p - 2, \text{ a constant for any given surface.}$$

Harkness and Morley, p. 229, and Forsyth, p. 317, give Neumann's proof. Riemann's proof will be found in his Gesammelte Werke, pp. 10, 11; also in Durège's Elemente der Theorie der Functionen, pp. 183–190. For Lippich's proof see Durège, pp. 190–197.

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July, 1897

# *Development of the A-process in Quaternions, with a Geometrical Application.*

BY JAMES BYRNIE SHAW, D. SC.

## I.—*Formulæ.*

1. Let us write (compare Hamilton, "Elements of Quaternions," p. 492, German edition, p. 741),

$$A.pq = \tfrac{1}{2}(pq - qp), \qquad (1)$$
$$V.A.pqr = -V(pAqr + qArp + rApq), \qquad (2)$$
$$S.A.pqr = S.pAqr. \qquad (3)$$

We have at once from these equations

$$A.pq = A.KpKq = V.VpVq = -A.qp, \qquad (4)$$
$$A.pp = 0. \qquad (5)$$
$$S.pAqr = S.VpVqVr = S.qArp = SrApq = -SqApr$$
$$= -SpArq = -SrApq, \qquad (6)$$
$$S.pApq = 0. \qquad (7)$$
$$A.pqr = S.pAqr - A.qrSp - ArpSq - ApqSr, \qquad (8)$$
$$= S.pAqr - V.pAqr - V.qArp - V.rApq. \qquad (9)$$
$$A.pqr = -A.qpr = A.qrp = -A.rqp = A.rpq = -A.prq, \qquad (10)$$
$$A.ppq = 0. \qquad (11)$$
$$S.pAqrs = Sp.SqArs - Sq.SrAsp + Sr.SsApq - Ss.SpAqr, \qquad (12)$$
$$= Sp.Aqrs - Sq.Arsp + Sr.Aspq - Ss.Apqr, \qquad (13)$$
$$= pS.qArs - qS.rAsp + rS.sApq - sSpAqr, \qquad (14)$$
$$= pAqrs - qArsp + rAspq - sApqr. \qquad (15)$$

$$
\left.\begin{aligned}
&S.pAqrs = -S.qArsp = \phantom{+}S.rAspq = -S.sApqr,\\
= -&S.pArqs = +S.rAqsp = -S.qAspr = +S.sAprq,\\
= \phantom{-}&S.pArsq = -S.rAsqp = +S.sAqpr = -S.qAprs,\\
= -&S.pAsrq = +S.sArqp = -S.rAqps = +S.qApsr,\\
= \phantom{-}&S.pAsqr = -S.sAqrp = +S.qArps = -S.rApsq,\\
= -&S.pAqsr = +S.qAsrp = -S.sArpq = +S.rApqs.
\end{aligned}\right\} \tag{16}
$$

2. These functions are visibly alternating, and are in nature determinantal. This aspect is more easily seen if we write

$$
\left.\begin{aligned}
p &= w_1 + x_1 i + y_1 j + z_1 k,\\
q &= w_2 + x_2 i + y_2 j + z_2 k,\\
r &= w_3 + x_3 i + y_3 j + z_3 k,\\
s &= w_4 + x_4 i + y_4 j + z_4 k,
\end{aligned}\right\} \tag{17}
$$

whence

$$A.rs = i\,|y_3 z_4| + j\,|z_3 x_4| + k\,|x_3 y_4|, \tag{18}$$

$$S.qArs = -\,|x_2 y_3 z_4|, \tag{19}$$

$$V.Aqrs = -\,i\,|w_2 y_3 z_4| - j\,|w_2 z_3 x_4| - k\,|w_2 x_3 y_4|, \tag{20}$$

$$S.Aqrs = -\,|x_2 y_3 z_4|, \tag{21}$$

$$S.pAqrs = -\,|w_1 x_2 y_3 z_4|. \tag{22}$$

3. Let
$$t = h_1 p + h_2 q + h_3 r + h_4 s, \tag{23}$$

wherein the $h$'s are scalars. Then we deduce easily

$$
\left.\begin{aligned}
S.tA.qrs &= +\,h_1 S.pAqrs,\\
S.tA.rsp &= -\,h_2 S.pAqrs,\\
S.tA.spq &= +\,h_3 S.pAqrs,\\
S.tA.pqr &= -\,h_4 S.pAqrs,
\end{aligned}\right\} \tag{24}
$$

therefore

$$t\,S.pAqrs = p\,S.tAqrs - q\,S.tArsp + r\,S.tAspq - s\,S.tApqr, \tag{25}$$

at least if $S.pAqrs \neq 0$. It is evident that this is the complete expression of any given quaternion $t$, and is a general relation holding between any five quaternions.

So also we have

$$t\,SpAqrs = -\,Sst.Apqr + Spt.Aqrs - Sqt.Arsp + Srt.Aspq. \tag{26}$$

4. We have from the definitions, if $a'$, $b'$, $x'$, etc., denote (as they will throughout the paper) scalars,

$$A(a'p + b'q)r = a'Apr + b'Aqr, \tag{27}$$
$$S.(a'p + b'q)Ars = a'SpArs + b'SqArs, \tag{28}$$
$$A.(a'p + b'q)rs = a'A.prs + b'A.qrs, \tag{29}$$
$$S.(a'p + b'q)Arst = a'S.pArst + b'S.qArst. \tag{30}$$

It is an easy corollary that if in any one of these functions, one of the quaternions is a linear scalar function of the others, the expression vanishes.

5. Conversely, if

$$A.pq = 0, \quad p = x'q + y', \tag{31}$$
$$S.pAqr = 0, \quad p = x'q + y'r + z', \tag{32}$$
$$A.pqr = 0, \quad p = x'q + y'r, \tag{33}$$
$$S.pAqrs = 0, \quad p = x'q + y'r + z's. \tag{34}$$

6. The following special cases deserve attention:

$$\left.\begin{array}{c} A.ij = k, \\ A.x'j = 0, \\ S.iAjk = -1, \quad S.x'Ajk = 0, \\ A.ijk = -1 = \omega, \quad A.jk\omega = i, \quad A.k\omega i = -j, \quad A.\omega ij = k, \\ S.\omega Aijk = 1. \end{array}\right\} \tag{35}$$

$$A.VpVq = A.pq, \quad A.SpVq = 0, \tag{36}$$
$$S.VpVqVr = SpAqr, \tag{37}$$
$$A.VpVqVr = SpAqr, \tag{38}$$
$$A.SpVqVr = -Sp.Aqr, \tag{39}$$
$$S.VpA.VqVrVs = 0, \tag{40}$$
$$S.SpA.VqVrVs = Sp.S.qArs. \tag{41}$$

7. Since

$$A.pqArst$$

is a quaternion, let us develop it in terms of $p$, $r$, $s$, $t$. Assume

$$A.pqArst = h_0 p + h_1 r + h_2 s + h_3 t.$$

Operating by $S.Arst$, we obtain at once

$$0 = h_0 SpArst, \tag{42}$$
$$\therefore\ h_0 = 0. \tag{43}$$

Operating by $S.p$ and $S.q$, we obtain

$$\left.\begin{aligned} 0 &= h_1 Spr + h_2 Sps + h_3 Spt, \\ 0 &= h_1 Sqr + h_2 Sqs + h_3 Sqt, \end{aligned}\right\} \tag{44}$$

$$\therefore\ h_1 : h_2 : h_3 = (Sps\, Sqt - Sqs\, Spt) : (Spt\, Sqr - Spr\, Sqt) : (Spr\, Sqs - Sps\, Sqr). \tag{45}$$

Hence if $h'$ is a yet undetermined scalar,

$$h'A.pqArst = \begin{vmatrix} r & s & t \\ Spr & Sps & Spt \\ Sqr & Sqs & Sqt \end{vmatrix}. \tag{46}$$

To determine $h'$, take scalars of each side, whence

$$h'S.Apq\, Arst = \begin{vmatrix} Sr & Ss & St \\ Spr & Sps & Spt \\ Sqr & Sqs & Sqt \end{vmatrix} \tag{47}$$

$$= \begin{vmatrix} Sr & Ss & St \\ S.VpVr & S.VpVs & S.VpVt \\ S.VqVr & S.VqVs & S.VqVt \end{vmatrix} \tag{48}$$

$$= - Sr.S.VpVqV.VsVt - Ss.S.VpVqV.VtVr - St.S.VpVqV.VrVs \tag{49}$$

$$= S.VpVqVArst = S.A.pqArst, \tag{50}$$

$$\therefore\ h' = 1$$

and

$$A.pqArst = \begin{vmatrix} r & s & t \\ S.rp & S.sp & S.tp \\ S.rq & S.sq & S.tq \end{vmatrix}. \tag{51}$$

This formula is strictly analogous to the vector formula

$$V.\alpha V\beta\gamma = -\begin{vmatrix} \beta & \gamma \\ S\beta\alpha & S\gamma\alpha \end{vmatrix}.$$

This result might have been obtained likewise by direct expansion:

$$\begin{aligned} A.pqA.rst &= S.ApqA.rst + V.ApqA.rst \\ &= - S.VpVq(Sr.V.VsVt + Ss.V.VtVr + St.V.VrVs) \\ &\quad - S.VrVsVt.V.VpVq \\ &\quad + Sp.V.Vq(Sr.V.VsVt + Ss.V.VtVr + St.V.VrVs) \\ &\quad - Sq.V.Vp(Sr.V.VsVt + Ss.V.VtVr + St.V.VrVs). \end{aligned}$$

Now the second line is equal to

$$-Vr.S.V.VsVt.V.VpVq - Vs.S.V.VtVrV.VpVq - Vt.S.V.VrVsV.VpVq.$$

The last two combine into

$$\begin{aligned} Vr.(&+ Sp.Ss.S.VqVt - Sp.St.S.VqVs \\ &- Sq.Ss.S.VpVt + Sq.St.S.VpVs) \\ &+ \text{etc.}, \end{aligned}$$

which, added to the reduced expression preceding, is

$$= \begin{vmatrix} Vr & Vs & Vt \\ S.pr & S.ps & S.pt \\ S.qr & S.qs & S.qt \end{vmatrix}.$$

The first line is easily seen to be the determinant

$$\begin{vmatrix} Sr & Ss & St \\ S.pr & S.ps & S.pt \\ S.qr & S.qs & S.qt \end{vmatrix}.$$

Whence

$$A.pqA.rst = \begin{vmatrix} r & s & t \\ Spr & Sps & Spt \\ Sqr & Sqs & Sqt \end{vmatrix}.$$

If we expand each side of the equality $A.pA.qrsA.stu = -A.pA.stuA.qrs$ by this theorem, we get

$$\begin{aligned} &Sps.(qS.sA.tur - rS.sA.tuq + tS.sA.qru - uS.sA.qrt) \\ &= S.(Spq.S.sA.tur - Spr.S.sA.tuq + S.pt.S.sA.qru - S.pu.S.sA.qrt). \end{aligned}$$

Since $p$ may be any quaternion, let it be $A.qrt$, whence

$$s.S.uA.qrt = uS.sA.qrt - qS.sA.rtu + rS.sA.tuq - tS.sA.uqr,$$

which is the deduction of this important formula analogous to Hamilton's deduction of the corresponding vector formula. From this formula, following Hamilton's process for vectors, we may deduce several others, showing relations between quaternions. Thus

$$A\,.\,pAqrs\,Auvw = \begin{vmatrix} u & v & w \\ SuAqrs & SvAqrs & SwAqrs \\ Spu & Spv & Spw \end{vmatrix} = -\begin{vmatrix} q & r & s \\ SqAuvw & SrAuvw & SsAuvw \\ Sqp & Srp & Ssp \end{vmatrix},$$

a relation connecting any six quaternions.

8. We have from (51)

$A\,.\,Auvw\,Axyz\,Arst$

$$= \begin{vmatrix} r & s & t \\ S.rAuvw & S.sAuvw & S.tAuvw \\ S.rAxyz & S.sAxyz & S.tAxyz \end{vmatrix} = \begin{vmatrix} u & v & w \\ S.uAxyz & S.vAxyz & S.wAxyz \\ S.uArst & S.vArst & S.wArst \end{vmatrix}$$

$$= \begin{vmatrix} x & y & z \\ S.xArst & S.yArst & S.zArst \\ S.xAuvw & S.yAuvw & S.zAuvw \end{vmatrix}. \qquad (52)$$

Operating on (51) by $S.u$

$$S.uApqArst = \begin{vmatrix} S.ru & S.su & S.tu \\ S.rp & S.sp & S.tp \\ S.rq & S.sq & S.tq \end{vmatrix} \qquad (53)$$

$$= S.rA.stAupq \qquad (54)$$

$$= -S(A.rst.Aupq), \qquad (55)$$

$$\therefore \quad -S^2.A.rst = \begin{vmatrix} S.r^2 & S.rs & S.rt \\ S.rs & S.s^2 & S.st \\ S.rt & S.st & S.t^2 \end{vmatrix}. \qquad (56)$$

Operating on (52) by $S.A.lmn$, we have

$$S.AlmnA.Auvw\,Axyz\,Arst = \begin{vmatrix} S.rAlmn & S.sAlmn & S.tAlmn \\ S.rAuvw & S.sAuvw & S.tAuvw \\ S.rAxyz & S.sAxyz & S.tAxyz \end{vmatrix}. \qquad (57)$$

From (58), making

$$u = m, \quad v = n, \quad w = p,$$
$$x = n, \quad y = p, \quad z = l,$$
$$r = p, \quad s = l, \quad t = m,$$

we have

$$S.Almn\,A.Amnp\,A.npl\,A.plm = (S.lAmnp)^3, \qquad (58)$$

and from (52) in like manner

$$A\,.\,Almn\;Amnp\;Anpl = -\,n\,S^2.\,lAmnp. \tag{59}$$

These two are strictly analogous to

$$V.\,V\alpha\beta\;V\beta\gamma = -\,\beta S\alpha\beta\gamma,$$
$$S\,.\,V\alpha\beta\;V\beta\gamma\;V\gamma\alpha = -\,S^2.\,\alpha\beta\gamma.$$

9. We deduce easily

$$S\,.\,(l+m)\,A\,(m+n)(n+p)(p+l) = 0, \tag{60}$$

$$S\,.\,(l+m+n)\,A\,.\,(m+n+p)(n+p+l)(p+l+m) = 3S\,.\,lAmnp, \tag{61}$$

$$\begin{aligned} S\,.\,A\,(l+m+n)(m+n+p)(n+p+l)\;A\,.\,A\,&(m+n+p)(n+p+l)(p+l+m)\\ A\,&(n+p+l)(p+l+m)(l+m+n)\\ A\,&(p+l+m)(l+m+n)(m+n+p)\\ &= 27S^3.\,lAmnp, \end{aligned} \tag{62}$$

$$\begin{aligned} S\,.\,A\,(l+m)(m+n)(n+p)\;A\,.\,A\,&(m+n)(n+p)(p+l)\\ A\,&(n+p)(p+l)(l+m)\\ A\,&(p+l)(l+m)(m+n) = 0, \end{aligned} \tag{63}$$

$$\begin{aligned} S\,.\,A\,.\,(A\,.\,lmn\,.\,A\,.\,mnp\,.\,A\,.\,npl)\;A\,.\,A\,.\,&(A\,.\,mnp\,.\,A\,.\,npl\,.\,A\,.\,plm)\\ A\,.\,&(A\,.\,npl\,.\,A\,.\,plm\,.\,A\,.\,lmn)\\ A\,.\,&(A\,.\,plm\,.\,A\,.\,lmn\,.\,A\,.\,mnp) = S^9.\,lAmnp. \end{aligned} \tag{64}$$

Formulas like these, of course, express determinant theorems, and might be written out in very elaborate form in that way. It is evident they can be developed indefinitely. It may be as well, however, to remark here, that as their vector analogs express theorems in spherical trigonometry, so these can be used as proofs of theorems in space of four dimensions analogous to those in the geometry of the sphere.

For any ten quaternions whatever

$$\begin{vmatrix} Spp_1 & Spq_1 & Spr_1 & Sps_1 & Spt_1 \\ Sqp_1 & Sqq_1 & Sqr_1 & Sqs_1 & Sqt_1 \\ Srp_1 & Srq_1 & Srr_1 & Srs_1 & Srt_1 \\ Ssp_1 & Ssq_1 & Ssr_1 & Sss_1 & Sst_1 \\ Stp_1 & Stq_1 & Str_1 & Sts_1 & Stt_1 \end{vmatrix} = 0. \tag{65}$$

For, if we multiply the first row by $S\,.\,qArst$, the second by $S\,.\,rAstp$, the third by $S.sAtpq$, the fourth by $S.tApqr$, the fifth by $S.pAqrs$, and then add the last four rows to the first, the first row will vanish.

Again, if for $Spp_1$ we write $Sp_0p_1$ the determinant reduces to

$$S.p_1(p_0-p)\begin{vmatrix} Sqq_1 & Sqr_1 & Sqs_1 & Sqt_1 \\ Srq_1 & Srr_1 & Srs_1 & Srt_1 \\ Ssq_1 & Ssr_1 & Sss_1 & Sst_1 \\ Stq_1 & Str_1 & Sts_1 & Stt_1 \end{vmatrix}, \tag{66}$$

which being resolved, and (53) applied, becomes

$$= Sp_1(p_0-p)(Sqq_1\, S.rAstAr_1s_1t_1 - Srq_1\, S.stAq\, Ar_1s_1t_1$$
$$+ S.sq_1\, S.tAqr\, Ar_1s_1t_1 - Stq_1 . S.tq\, Ars\, Ar_1s_1t_1) \tag{67}$$
$$= - S.p_1(p_0-p)\, S.q_1Ar_1s_1t_1\, S.qArst. \tag{68}$$

We have incidentally proved here that

$$\begin{vmatrix} Sqq_1 & Sqr_1 & Sqs_1 & Sqt_1 \\ Srq_1 & Srr_1 & Srs_1 & Srt_1 \\ Ssq_1 & Ssr_1 & Sss_1 & Sst_1 \\ Stq_1 & Str_1 & Sts_1 & Stt_1 \end{vmatrix} = - S.qArst\, S.q_1Ar_1s_1t_1, \tag{69}$$

and as a corollary

$$S^2.qArst = -\begin{vmatrix} S.q^2 & Sqr & Sqs & Sqt \\ S.qr & S.r^2 & Srs & Srt \\ S.qs & Srs & S.s^2 & Sst \\ S.qt & Srt & Sst & S.t^2 \end{vmatrix} \tag{70}$$

10. Let us consider the set of quaternions

$$\left.\begin{aligned} l_1 &= \tfrac{1}{2}\theta^{-1}(\theta+i+j+k), \\ l_2 &= \tfrac{1}{2}\theta^{-1}(\theta-i-j+k), \\ l_3 &= \tfrac{1}{2}\theta^{-1}(\theta-i+j-k), \\ l_4 &= \tfrac{1}{2}\theta^{-1}(\theta+i-j-k), \\ \theta &= \sqrt{-1}. \end{aligned}\right\} \tag{71}$$

We have from them at once the equations

$$\left.\begin{aligned} S\,.\,l_r^2 &= \theta\,, \\ S\,.\,l_r l_s &= 0\,. \end{aligned}\right\} \qquad (72)$$

$$A\,.\,l_1l_2l_3 = -\,l_4,\quad A\,.\,l_2l_3l_4 = l_1,\quad A\,.\,l_3l_4l_1 = -\,l_2,\quad A\,.\,l_4l_1l_2 = l_3. \qquad (73)$$

$$S\,.\,l_1Al_2l_3l_4 = \theta\,. \qquad (74)$$

The latter equation shows that they are linearly independent. The symmetry of the equations is apparent. These four bear a close analogy to the vector $i, j, k$.

11. The following formulæ are of use:

$$\left.\begin{aligned} \theta &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 + l_2 + l_3 + l_4), \\ i &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 - l_2 - l_3 + l_4), \\ j &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 - l_2 + l_3 - l_4), \\ k &= \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 + l_2 - l_3 - l_4), \end{aligned}\right\} \qquad (75)$$

$$\left.\begin{aligned} \theta^{\frac{1}{2}} &= \tfrac{1}{2}(l_1 + l_2 + l_3 + l_4), \\ \theta &= -\,1 + \tfrac{1}{2}\sqrt{2}(l_1 + l_2 + l_3 + l_4). \end{aligned}\right\} \qquad (76)$$

$$\left.\begin{aligned} 4\theta &= l_1^2 + l_2^2 + l_3^2 + l_4^2, \\ i &= l_1^2 - l_2^2 - l_3^2 + l_4^2, \\ j &= l_1^2 - l_2^2 + l_3^2 - l_4^2, \\ k &= l_1^2 + l_2^2 - l_3^2 - l_4^2. \end{aligned}\right\} \qquad (77)$$

$$\left.\begin{aligned} l_1^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(7l_1 + 3l_2 + 3l_3 + 3l_4), \\ l_2^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(3l_1 + 7l_2 + 3l_3 + 3l_4), \\ l_3^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(3l_1 + 3l_2 + 7l_3 + 3l_4), \\ l_4^2 &= \tfrac{1}{8}\theta^{\frac{1}{2}}(3l_1 + 3l_2 + 3l_3 + 7l_4). \end{aligned}\right\} \qquad (78)$$

$$\left.\begin{aligned} A\,.\,l_1l_2 &= \tfrac{1}{2}\theta(-\,i + j) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_3 - l_4), \\ A\,.\,l_1l_3 &= \tfrac{1}{2}\theta(\;\;\,i - k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_4 - l_2), \\ A\,.\,l_4l_1 &= \tfrac{1}{2}\theta(\;\;\,j - k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_3 - l_2), \\ A\,.\,l_2l_3 &= \tfrac{1}{2}\theta(\;\;\,j + k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_1 - l_4), \\ A\,.\,l_2l_4 &= \tfrac{1}{2}\theta(-\,i - k) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_3 - l_1), \\ A\,.\,l_3l_4 &= \tfrac{1}{2}\theta(\;\;\,i + j) = \tfrac{1}{2}\theta^{\frac{3}{2}}(l_1 - l_2). \end{aligned}\right\} \qquad (79)$$

$$\left.\begin{aligned} V.l_1 &= \tfrac{1}{4}(\ \ 3l_1 - \ \ l_2 - \ \ l_3 - \ \ l_4), \\ V.l_2 &= \tfrac{1}{4}(-l_1 + 3l_2 - \ \ l_3 - \ \ l_4), \\ V.l_3 &= \tfrac{1}{4}(-l_1 - \ \ l_2 + 3l_3 - \ \ l_4), \\ V.l_4 &= \tfrac{1}{4}(-l_1 - \ \ l_2 - \ \ l_3 + 3l_4). \end{aligned}\right\} \qquad (80)$$

12. Since $l_1$, $l_2$, $l_3$, $l_4$ are linearly independent, any other quaternion $q$ may be expressed in terms of them. Suppose then

$$q = h'l_1 + h''l_2 + h'''l_3 + h^{\text{IV}}l_4. \qquad (81)$$

We have at once

$$h^{(r)} = -\theta . S.l_r q, \qquad (82)$$

$$S.q = \tfrac{1}{2}\theta^{\frac{1}{2}}(h' + h'' + h''' + h^{\text{IV}}), \qquad (83)$$

$$S.q^2 = \theta(h'^2 + h''^2 + h'''^2 + h^{\text{IV}2}), \qquad (84)$$

$$Vq = \tfrac{1}{4}l_1(3h' - h'' - h''' - h^{\text{IV}}) + \tfrac{1}{4}l_2(-h' + 3h'' - h''' - h^{\text{IV}})$$
$$+ \tfrac{1}{4}l_3(-h' - h'' + 3h''' - h^{\text{IV}}) + \tfrac{1}{4}l_4(-h' - h'' - h''' + 3h^{\text{IV}}). \qquad (85)$$

13. If

$$r = \Sigma . g'l_1, \qquad (85)$$

$$S.qr = \theta\Sigma . h'g', \qquad (86)$$

$$A.qr = A.l_1l_2|h'g''| + \text{etc.} \qquad (87)$$

So also

$$A.qrs = l_1|h''g'''f^{\text{IV}}| - l_2|h'''g^{\text{IV}}f'| + l_3|h^{\text{IV}}g'f''| - l_4|h'g''f'''|. \qquad (88)$$

$$S.qArst = \theta.|h'g''f'''e^{\text{IV}}|. \qquad (89)$$

14. We solve the linear quaternion operator by these functions thus. Let

$$r = \Phi . q. \qquad (90)$$

Let $s$, $t$, $u$ be any quaternions satisfying the equation

$$\Phi q = A.stu, \qquad (91)$$

$$\therefore\ S.s\Phi q = 0,\ \ S.t\Phi q = 0,\ \ S.u\Phi q = 0. \qquad (92)$$

Now these three may also be written

$$S.q\Phi's = 0,\ \ S.q\Phi't = 0,\ \ S.q\Phi'u = 0, \qquad (93)$$

$$\therefore\ mq = A.\Phi's\,\Phi't\,\Phi'u. \qquad (94)$$

Operate by $S.\Phi'v$, $\therefore$

$$mS.q\Phi'v = S.\Phi'v\,A.\Phi's\,\Phi't\,\Phi'u. \qquad (95)$$

Whence
$$m=\frac{S.\Phi's\,A\Phi't\,\Phi'u\,\Phi'v}{S.sAtuv}. \tag{96}$$

In this equation it is evident that $s$, $t$, $u$, $v$ may be *any* linearly independent quaternions. For if for either we substitute an arbitrary linear function of all four, we find the expression reducing to exactly the same again as (96).

Put $\Phi - g$ for $\Phi$, whence

$$m_g q = m_g(\Phi - g)^{-1}A.stu \tag{97}$$
$$= A(\Phi' - g)s(\Phi' - g)t(\Phi' - g)u, \tag{98}$$
$$= A.\Phi's\,\Phi't\,\Phi'u - g(A.s\Phi't\,\Phi'u + A.t\Phi'u\,\Phi's + A.u\Phi's\,\Phi't)$$
$$+ g^2(A.st\Phi'u + A.tu\Phi's + A.us\Phi't) - g^3A.stu. \tag{99}$$

Here
$$m_g = m - m_1g + m_2g^2 - m_3g^3 + g^4, \tag{100}$$
where

$$m_1 = \frac{S.sA\Phi't\Phi'u\Phi'v + S.\Phi'sA.t\Phi'u\Phi'v + S.\Phi'sA\Phi'tu\Phi'v + S.\Phi'sA\Phi't\Phi'uv}{S.sAtuv}, \tag{101}$$

$$m_2 = \frac{\begin{array}{l}S.sAt\Phi'u\Phi'v + S.sA\Phi'tu\Phi'v + S.sA.\Phi't\Phi'uv \\ \quad + S.\Phi'sA.tu\Phi'v + S.\Phi'sA.t\Phi'uv + S.\Phi'sA\Phi'tuv\end{array}}{S.sAtuv}, \tag{102}$$

$$m_3 = \frac{S.sAtu\Phi'v + S.sAt\Phi'uv + S.sA\Phi'tuv + S.\Phi'sAtuv}{S.sAtuv}. \tag{103}$$

Substitute these values, operate by $\Phi - g$ and equate like powers of $g$:

$$m\,A.stu = \Phi A.\Phi's\,\Phi't\,\Phi'u, \tag{104}$$
$$m_1A.stu = \Phi(A.s\Phi't\,\Phi'u + A.t\Phi'u\,\Phi's + A.u\Phi's\,\Phi't) + A.\Phi's\,\Phi't\,\Phi'u, \tag{105}$$
$$m_2A.stu = \Phi(A.st\Phi'u + A.tu\Phi's + A.us\Phi't)$$
$$+ (A.s\Phi't\,\Phi'u + A.t\Phi'u\,\Phi's + A.u\Phi's\,\Phi't), \tag{106}$$
$$m_3A.stu = \Phi(A.stu) + (A.st\Phi'u + A.tu\Phi's + A.us\Phi't). \tag{107}$$

It is plain from (104) and (105) that the parenthesis in (105) is a function of $A.stu$, say
$$\mathrm{X}.Astu = (A.s\Phi't\,\Phi'u + A.t\Phi'u\,\Phi's + A.u\Phi's\,\Phi't). \tag{108}$$

Likewise we may write

$$\Psi A.stu = A.st\Phi'u + A.tu\Phi's + A.us\Phi't; \tag{109}$$
$$\therefore\ m_1 = \Phi\mathrm{X} + m\Phi^{-1}, \tag{110}$$

$$m_2 = \Phi\Psi + X, \tag{111}$$
$$m_3 = \Phi + \Psi, \tag{112}$$

whence

$$\Psi = m_3 - \Phi, \tag{113}$$
$$X = m_2 - m_3\Phi + \Phi^2, \tag{114}$$
$$m\Phi^{-1} = m_1 - m_2\Phi + m_3\Phi^2 - \Phi^3, \tag{115}$$
$$\Phi^4 - m_3\Phi^3 + m_2\Phi^2 - m_1\Phi + m = 0. \tag{116}$$

15. If $m_3$ vanish, (107) gives

$$\Phi . Astu = -(A . st\Phi'u + A . tu\Phi's + A . us\Phi't). \tag{117}$$

If $m_2$ vanish, (106) gives

$$\Phi(A . st\Phi'u + A . tu\Phi's + A . us\Phi't) = -(A . s\Phi't\Phi'u + A . t\Phi'u\Phi's + A . u\Phi's\Phi't). \tag{118}$$

If $m_1$ vanish,

$$A . \Phi's\Phi't\Phi'u = \Phi(A . s\Phi't\Phi'u + A . t\Phi'u\Phi's + A . u\Phi's\Phi't). \tag{119}$$

If $m$ vanish,

$$\Phi A . \Phi's\Phi't\Phi'u = 0. \tag{120}$$

If $m$ and $m_1$ vanish, we have in addition to (120),

$$\Phi^2(A . s\Phi't\Phi'u + A . t\Phi'u\Phi's + A . u\Phi's\Phi't) = 0. \tag{121}$$

If also $m_2$ vanishes, we may add the equation

$$\Phi^3(A . st\Phi'u + A . tu\Phi's + A . us\Phi't) = 0. \tag{122}$$

And finally, if $m_3$ is also zero,

$$\Phi^4 . A . stu = 0. \tag{123}$$

16. If $\Phi$ is self-transverse, it may be written

$$\Phi = -g_1\theta l_1 Sl_1 + g_2\theta l_2 Sl_2 - g_3\theta l_3 Sl_3 + g_4\theta l_4 Sl_4, \tag{124}$$

where $l_1, l_2, l_3, l_4$ are as in §10.

For let $q_1, q_2, q_3, q_4$ be the axes of $\Phi$, then

$$\Phi q_1 = g_1 q_1, \text{ etc.}, \tag{125}$$

$$\left.\begin{aligned} \therefore\ S . q_2\Phi q_1 &= g_1 Sq_1q_2, \\ S . q_1\Phi q_2 &= g_2 Sq_1q_2, \end{aligned}\right\} \tag{126}$$

$$\therefore\ (g_1 - g_2) S . q_1q_2 = 0, \tag{127}$$

and if $g_1, g_2, g_3, g_4$ are all different,

$$S.q_1q_2 = 0, \text{ etc.} \tag{128}$$

Further,

$$A.\Phi q_1\Phi q_2\Phi q_3 = m\Phi^{-1}A.q_1q_2q_3 = g_1g_2g_3Aq_1q_2q_3 = g_1g_2g_3g_4\Phi^{-1}A.q_1q_2q_3, \tag{129}$$

$$\therefore\ \Phi Aq_1q_2q_3 = g_4Aq_1q_2q_3 = g_4q_4. \tag{130}$$

Likewise

$$\Phi A.q_2q_3q_4 = g_1A.q_2q_3q_4 = -g_1(-q_1). \tag{131}$$

Hence the axes satisfy such relations as those defining the $l$'s of §10.

But we may also write $\Phi$ in the form

$$\Phi = g_1S.1 - g_2iSi - g_3jSj - g_4kSk, \tag{132}$$

since

$$S.1i = Sij = 0, \text{ etc.} \tag{133}$$

and

$$\left.\begin{aligned} \Phi.A1ij &= -g_4k, & -k &= A.1ij, \\ \Phi.Aijk &= -g_11, & -1 &= A.ijk, \\ \Phi.Ajk1 &= -g_2i, & -i &= A.jk1, \\ \Phi.A.k1i &= -g_3j, & -j &= A.k1i. \end{aligned}\right\} \tag{134}$$

17. If some of the roots are equal, results hold analogous to those in the theory of the linear vector operator by much the same proofs.

18. The general expression for $\Phi$ in any case, in terms of its roots and axes, is

$$\Phi = \frac{g_1q_1S()A.q_2q_3q_4 - g_2q_2S()Aq_3q_4q_1 + g_3q_3S()Aq_4q_1q_2 - g_3q_4S()Aq_1q_2q_3}{S.q_1Aq_2q_3q_4}, \tag{135}$$

whence $\Phi'$ may be written

$$\Phi' = \frac{\begin{aligned} &g_1A.q_2q_3q_4Sq_1() - g_2A.q_3q_4q_1S.q_2() \\ &\qquad + g_3A.q_4q_1q_2S.q_3() - g_4A.q_1q_2q_3Sq_4() \end{aligned}}{S.q_1Aq_2q_3q_4}. \tag{136}$$

19. If we write

$$\left.\begin{aligned} K_1 &= \frac{q_1S()Aq_2q_3q_4}{S.q_1Aq_2q_3q_4}, \\ K_2 &= -\frac{q_2S()Aq_3q_4q_1}{S.q_1Aq_2q_3q_4}, \\ K_3 &= \frac{q_3S()Aq_4q_1q_2}{S.q_1Aq_2q_3q_4}, \\ K_4 &= -\frac{q_4S()Aq_1q_2q_3}{S.q_1Aq_2q_3q_4}, \end{aligned}\right\} \tag{137}$$

whence $$K_r^2 = K_r, \quad K_rK_s = 0; \tag{138}$$

and $$I = K_1 + \theta K_2 - K_3 - \theta K_4, \tag{139}$$

whence $$I^2 = K_1 - K_2 + K_3 - K_4, \tag{140}$$
$$I^3 = K_1 - \theta K_2 - K_3 + \theta K_4, \tag{141}$$
$$I^4 = K_1 + K_2 + K_3 + K_4 = 1; \tag{142}$$

we have $$\Phi = g_1K_1 + g_2K_2 + g_3K_3 + g_4K_4, \tag{143}$$
$$= a + bI + cI^2 + dI^3. \tag{144}$$

In this equation

$$\left.\begin{aligned} a &= \tfrac{1}{4}(g_1 + g_2 + g_3 + g_4), \\ b &= \tfrac{1}{4}(g_1 - \theta g_2 - g_3 + \theta g_4), \\ c &= \tfrac{1}{4}(g_1 - g_2 + g_3 - g_4), \\ d &= \tfrac{1}{4}(g_1 + \theta g_2 - g_3 - \theta g_4). \end{aligned}\right\} \tag{145}$$

The general theory of these operators may thence be developed in the same manner as I have developed that of the linear *vector* operator (read before the American Math. Soc., Aug. 1895). The quaternion operator may also be written in a sedenion form, and many theorems thence deduced.

20. In the equation
$$m\Phi^{-1}A\,.\,stu = A\,.\,\Phi's\,\Phi't\,\Phi'u,$$

substitute $s$ for $\Phi's$, $\Phi'^{-1}s$ for $s$, etc., then change $\Phi$ to $\Phi - g$. We have
$$m_gA\,.\,(\Phi' - g)^{-1}s\,(\Phi' - g)\,t\,(\Phi' - g)\,u = (\Phi - g)\,A\,.\,stu. \tag{146}$$

Since

$$\begin{aligned} m_g(\Phi - g)^{-1}A\,.\,pqr &= A\,.\,(\Phi' - g)\,p\,(\Phi' - g)\,q\,(\Phi' - g)\,r \\ &= (m\Phi^{-1} - g\mathrm{X} + g^2\Psi - g^3)\,A\,.\,pqr, \end{aligned} \tag{147}$$

$$\left.\begin{aligned} \therefore\ \frac{m_g}{g}S.q\,(\Phi - g)^{-1}q &= \frac{m}{g}S.q\Phi^{-1}q - Sq\mathrm{X}q + gSq\Psi q - g^2S.q^2, \\ \frac{m_h}{h}S.q\,(\Phi - h)^{-1}q &= \frac{m}{h}S.q\Phi^{-1}q - Sq\mathrm{X}q + hSq\Psi q - h^2S.q^2, \end{aligned}\right\} \tag{148}$$

$$\begin{aligned} \therefore\ &\frac{m_g}{g}S.q\,(\Phi - g)^{-1}q - \frac{m_h}{h}S.q\,(\Phi - h)^{-1}q \\ &= (g - h)\left\{S.q\Psi q - (g + h)\,S.q^2 - \frac{mS.q\Phi^{-1}q}{gh}\right\}. \end{aligned} \tag{149}$$

These are identical when $g = h$, and also when

$$ghSq\Psi q - gh(g+h)S.q^3 - mSq\Phi^{-1}q = 0. \qquad (150)$$

Thus whatever $g$ or $h$, they are identical if

$$S.q\Psi q = 0,\ S.q^3 = 0, \text{ and } mSq\Phi^{-1}q = 0. \qquad (151)$$

These equations are of use in the theory of surfaces of the second order.

21. Again, if

$$\left.\begin{array}{l} S.q(\Phi - g)^{-1}q = 0, \\ S.q(\Phi - h)^{-1}q = 0, \end{array}\right\} \qquad (152)$$

$$\therefore \left.\begin{array}{l} mSq\Phi^{-1}q - gSqXq + g^2Sq\Psi q - g^3S.q^3 = 0, \\ mSq\Phi^{-1}q - hSqXq + h^2Sq\Psi q - h^3S.q^3 = 0. \end{array}\right\} \qquad (153)$$

Hence whatever $x$,

$$m(1-x)Sq\Phi^{-1}q - (g - hx)SqXq + (g^2 - h^2x)Sq\Psi q - (g^3 - h^3x)Sq^3 = 0. \quad (154)$$

If $x = 1$, $\quad SqXq + (g+h)Sq\Psi q - (g^2 + gh + h^2)Sq^3 = 0. \qquad (155)$

If $x = g/h$, $\quad mSq\Phi^{-1}q - gSq\Psi q + g(g+h)S.q^3 = 0. \qquad (156)$

If $x = g^2/h^2$, $\quad m(g+h)Sq\Phi^{-1}q - ghSqXq - g^2h^2S.q^3 = 0. \qquad (157)$

If $x = g^3/h^3$, $\quad m(g^2 + gh + h^2)Sq\Phi^{-1}q - gh(g+h)SqXq + g^2h^2Sq\Psi q = 0. \quad (158)$

22. Passing now to considerations of a different kind, let

$$q = x'l_1 + x''l_2 + x'''l_3 + x^{\text{IV}}l_4. \qquad (159)$$

Let

$$\Box = l_1\frac{\partial}{\partial x'} + l_2\frac{\partial}{\partial x''} + l_3\frac{\partial}{\partial x'''} + l_4\frac{\partial}{\partial x^{\text{IV}}}. \qquad (160)$$

Then we have

$$\begin{array}{lll}
\Box.q & = 4\theta, & (161) \\
\Box.Sq & = \tfrac{1}{2}\theta^{\frac{1}{2}}(l_1 + l_2 + l_3 + l_4) = \theta, & (162) \\
\Box.Vq & = 3\theta, & (163) \\
\Box.T^2q & = 2\theta Sq - 6\theta Vq, & (164) \\
\Box.Saq & = \theta a. & (165) \\
\Box.aq & = 4\theta Sa, & (166) \\
\Box.Vaq & = \theta(-3Sa + Va), & (167) \\
\Box.A.aq & = -2\theta.Va, & (168)
\end{array}$$

etc.

II.—*A Geometrical Application of the use of the A-Functions.*

1. Let $ABCD$ be any four non-coplanar points in space. Let us denote any point in space, as $P$, by the quaternion

$$p = \frac{x'l_1 + x''l_2 + x'''l_3 + x^{\mathrm{IV}}l_4}{\Sigma . x'}, \tag{1}$$

wherein

$$x' : x'' : x''' : x^{\mathrm{IV}} = (P\text{–}BCD) : (P\text{–}CDA) : (P\text{–}DAB) : (P\text{–}ABC). \tag{2}$$

2. Evidently $A$, $B$, $C$, $D$ are denoted respectively by

$$l_1, l_2, l_3, l_4.$$

3. The areas of the faces opposite $A$, $B$, $C$, $D$, being denoted by $a'$, $b'$, $c'$, $d'$, the perpendiculars from $P$ on the faces are evidently as

$$\frac{x'}{a'} : \frac{x''}{b'} : \frac{x'''}{c'} : \frac{x^{\mathrm{IV}}}{d'}.$$

4. If $Q$ be any second point $(y', y'', y''', y^{\mathrm{IV}})$, any point $R$ on the line joining $P$ and $Q$ is

$$r = \frac{m'p + n'q}{m' + n'}, \text{ where } \overline{PR} : \overline{RQ} = n' : m'.$$

The proof is easy by considering the relations of the perpendiculars involved.

This equation is exactly equivalent to

$$A . pqr = 0, \tag{3}$$

which is therefore the equation of the right line through $p$ and $q$.

5. The centroid $G$ of the fundamental tetrahedron is

$$\tfrac{1}{4}(l_1 + l_2 + l_3 + l_4) = g. \tag{4}$$

6. If $q$ be any point,

$$q = Sq + Vq. \tag{5}$$

But

$$Sq = (x' + x'' + x''' + x^{\mathrm{IV}})\, g, \tag{6}$$

hence

$$\Sigma x' = \frac{S . q}{g}. \tag{7}$$

When $S.q=0$,

$$\Sigma x' = 0 \tag{8}$$

and the point $Vq$ is at infinity. Hence $Vq$ is the point at infinity on the line joining $G$ and $Q$.

7. $$Kq = 2Sq - q = 2g - q \tag{9}$$

is evidently such a point that $G$ bisects the segment joining it to $Q$.

8. If three points $a_1$, $a_2$, $a_3$ be given, any point in the plane joining them is

$$p = x'a_1 + y'a_2 + z'a_3. \tag{10}$$

This is exactly equivalent to

$$S.pAa_1a_2a_3 = 0, \tag{11}$$

which is therefore the equation of the plane through $a_1$, $a_2$, $a_3$.

9. It is easy to see that, if lines be drawn from $A$, $B$, $C$, $D$ through a point $P$, to cut the opposite faces in $P_A$, $P_B$, $P_C$, $P_D$, and then lines be drawn through each of these points and the three vertices in the same plane with it to intersect the edges in $P_{AB}$, $P_{AC}$, $P_{AD}$, etc., then

$$\overline{BP}_{AD} : \overline{P_{AD}C} = x''' : x'', \text{ etc.,}$$

and $P_{AD} = x''l_2 + x'''l_3$.

10. If we define as the isotomic conjugate of $P(p = x'l_1 + x''l_2 + x'''l_3 + x^{\text{IV}}l_4)$ the point ${}_iP$,

$${}_iP = \frac{1}{x'}l_1 + \frac{1}{x''}l_2 + \frac{1}{x'''}l_3 + \frac{1}{x^{\text{IV}}}l_4, \tag{12}$$

then the harmonic conjugates of the points ${}_iP_{AB}$, ${}_iP_{AC}$, ${}_iP_{AD}$, ${}_iP_{BC}$, ${}_iP_{BD}$, ${}_iP_{CD}$ are represented respectively by $\frac{1}{x'''}l_3 - \frac{1}{x^{\text{IV}}}l_4$, $\frac{1}{x''}l_2 - \frac{1}{x^{\text{IV}}}l_4$,

$$\frac{1}{x''}l_2 - \frac{1}{x'''}l_3,\ \frac{1}{x'}l_1 - \frac{1}{x^{\text{IV}}}l_4,\ \frac{1}{x'}l_1 - \frac{1}{x'''}l_3,\ \frac{1}{x'}l_1 - \frac{1}{x''}l_2.$$

It is plain at a glance that any four of these lie in one plane, that is, all of them lie in a plane, namely, the plane

$$S.r(x'l_1 + x''l_2 + x'''l_3 + x^{\text{IV}}l_4) = 0. \tag{13}$$

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This plane is the polar plane of the point ${}_{\iota}P$ with respect to the tetrahedron $ABCD$, and we may call it the *isopolar* plane of $P$,

$$x'l_1 + x''l_2 + x'''l_3 + x^{\text{IV}}l_4 .$$

Hence $A.a_1a_2a_3$ is the isopole of the plane through $a_1$, $a_2$, $a_3$.

10. Let the points $a_1$, $a_2$, $a_3$ be collinear. Then any point on the intersection of

$$\left.\begin{array}{l} S.pa_1 = 0 , \\ S.pa_2 = 0 \end{array}\right\} \tag{14}$$

is satisfied by the points on the plane $S.pa_3 = 0$, and either of these two, for $a_3 = m'a_1 + n'a_2$. Hence if the isopoles of three planes are collinear, they are also collinear.

11. The point of intersection of the three planes

$$Spa_1 = 0 , \quad Spa_2 = 0 , \quad Spl_1 = 0$$

is

$$p = A.l_1a_1a_2 . \tag{15}$$

That of intersection of

$$Spa_1 = 0 , \quad Spa_2 = 0 , \quad Spl_2 = 0 ,$$

is

$$p = A.l_2a_1a_2 . \tag{16}$$

Hence the line of intersection of these two planes is

$$A.rAl_1a_1a_2\,A.l_2a_1a_2 = 0 , \tag{17}$$

and for $l_1$, $l_2$ may be written any two points in space. Any point on this line is

$$p = A.a_1a_2a_x , \tag{18}$$

where $a_x$ is any point in space.

12. The line of intersection of the isopolars of points on this line is

$$A.p\,(A.lAl_1a_1a_2\,Al_2a_1a_2)(A.l_3Al_1a_xa_2Al_2a_1a_2) = 0 = A.pa_1a_2 .$$

Any point on this latter line is then

$$p = A.a_x\,Al_1a_1a_2\,Al_2a_1a_2 , \tag{19}$$

where $a_x$ is any point in space.

Hence if there is given any line,

$$A.ra_1a_2 = 0 ,$$

there is a second line

$$A\,.\,pA\,.\,l_r a_1 a_2\,Al_s a_1 a_2 = 0, \qquad (20)$$

which may be called its isopolar, each possessing the property that all planes through it have their isopoles on the other.

If two lines are isopolars, as

$$A\,.\,ra_1a_2 = 0,$$
$$A\,.\,ra_3a_4 = 0,$$

then for the first any point

$$r = x'a_1 + y'a_2.$$

Hence

$$\left.\begin{array}{l} x'Sa_1a_3 + y'Sa_2a_3 = 0, \\ x'Sa_1a_4 + y'Sa_2a_4 = 0, \end{array}\right\} \qquad (21)$$

$$\therefore\ (Sa_1a_3\,Sa_2a_4 - Sa_1a_4\,Sa_2a_3) = 0$$
$$= S\,.\,Va_1\,Va_3\,SVa_2\,Va_4 - S\,.\,Va_1\,Va_4SVa_2\,Va_3 = S\,.\,Aa_1a_2\,Aa_3a_4. \qquad (22)$$

This is the condition of line-isopolarity.

The vector $A\,.\,a_1a_2$ I shall call the *Pluecker* of the line through $a_1$, $a_2$. For, the six coefficients in the expression

$$A\,.\,a_1a_2 = Al_1l_2\,|a_1'a_2''| + A\,.\,l_1l_3\,|a_1'a_2'''| + A\,.\,l_1l_4\,|a_1'a_2^{\text{IV}}|$$
$$+ A\,.\,l_2l_3\,|a_1''a_2'''| + A\,.\,l_2l_4\,|a_1''a_2^{\text{IV}}| + A\,.\,l_3l_4\,|a_1'''a_2^{\text{IV}}|$$

are Pluecker's coordinates of a line. They are connected by the relation

$$|a_1'a_2''|\,.\,|a_1'''a_2^{\text{IV}}| - |a_1'a_2'''|\,.\,|a_1''a_2^{\text{IV}}| + |a_1'a_2^{\text{IV}}|\,.\,|a_1''a_2'''| = 0.$$

Evidently

$$S\,.\,l_1Aa_1a_2 = \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(\quad P_{23} + P_{24} + P_{34}),$$
$$S\,.\,l_2Aa_1a_2 = \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(- P_{13} - P_{14} + P_{34}),$$
$$S\,.\,l_3Aa_1a_2 = \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(\quad P_{12} - P_{14} - P_{24}),$$
$$S\,.\,l_4Aa_1a_2 = \tfrac{1}{2}\,\theta^{\frac{1}{2}}\,(\quad P_{12} + P_{13} + P_{23}),$$
$$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0.$$

These five equations and the fact that only the ratios of the $p$'s is needed, give sufficient data for their determination.

Evidently $A\,.\,a_1a_2$ is on the line

$$A\,.\,l_ra_1a_2,\quad A\,.\,l_sa_1a_2.$$

The theory of complexes and congruences is developable from these equations.

13. The intersection of

$$Sp\alpha_1 = 0,$$
$$Sp\alpha_2 = 0,$$
$$Sp\alpha_3 = 0$$

is

$$p = A \,.\, \alpha_1\alpha_2\alpha_3. \tag{23}$$

Hence the point of intersection of three planes is the isopole of the plane of their isopoles.

14. The plane $S\,.\,gp = Sp = 0$

is the plane at infinity, since it is the polar as well as the isopolar of the centroid of the tetrahedron.

Two planes are parallel when they intersect only on the plane at infinity. Hence if they are $S\,.\,\alpha_1 p = 0$ and $S\alpha_2 p = 0$,

$$S\,.\,g\,(x' A\,.\,\alpha_y\alpha_1\alpha_2 + y' A\,.\,\alpha_z\alpha_1\alpha_2) = 0, \tag{24}$$

where $x'$ and $y'$ have any values and $\alpha_y$, $\alpha_z$ are any points. This is satisfied only by

$$\left.\begin{aligned} S\,.\,A\,.\,\alpha_y\alpha_1\alpha_2 = 0, \\ S\,.\,A\,.\,\alpha_z\alpha_1\alpha_2 = 0, \end{aligned}\right\} \tag{25}$$

and since $\alpha_y$ and $\alpha_z$ are arbitrary, it follows that

$$A\,.\,\alpha_1\alpha_2 = 0. \tag{26}$$

This is the condition of parallelity of the planes, viz. that the Pluecker of the line joining their isopoles be zero. It also gives

$$\alpha_1 = x'\alpha_2 + y'g. \tag{27}$$

Hence if two planes are parallel their isopoles are collinear with the centroid.

15. Two lines $A\,.\,p\alpha_1\alpha_2 = 0$ and $A\,.\,p\alpha_3\alpha_4 = 0$

intersect if

$$S\,.\,\alpha_1 A\alpha_2\alpha_3\alpha_4 = 0. \tag{28}$$

If $p$ is the point of intersection,

$$p = x\alpha_1 + y\alpha_2,$$
$$\therefore\ xA\,.\,\alpha_1\alpha_3\alpha_4 + yA\alpha_2\alpha_3\alpha_4 = 0,$$
$$\therefore\ x : y = S\alpha_2 A\alpha_3\alpha_4 : \ - S\alpha_1 A\alpha_3\alpha_4, \tag{29}$$
$$\left.\begin{aligned} \therefore\quad p &= \alpha_1 S\alpha_2\, A\alpha_3\alpha_4 - \alpha_2 S\alpha_1\, A\alpha_3\alpha_4 \\ &= \alpha_3 S\alpha_4\, A\alpha_1\alpha_2 - \alpha_4 S\alpha_3\, A\alpha_1\alpha_2. \end{aligned}\right\} \tag{30}$$

They are parallel if $Sp = 0$, i. e. if

$$\left.\begin{array}{l} Sa_1 \,.\, S\,.\,a_2 Aa_3 a_4 - Sa_2 \,.\, S\,.\,a_1 Aa_3 a_4 = 0\,, \\ \text{or} \quad Sa_3 \,.\, S\,.\,a_4 Aa_1 a_2 - Sa_4 \,.\, S\,.\,a_3 Aa_1 a_2 = 0\,. \end{array}\right\} \tag{31}$$

16. The perpendiculars let fall upon the plane $S\,.\,ap = 0$ from the vertices $l_1$, $l_2$, $l_3$, $l_4$ are as

$$S\,.\,al_1 : Sal_2 : Sal_3 : Sal_4. \tag{32}$$

For, these perpendiculars are parallel, hence we have a series of similar triangles which enable us to write out the relations at once.

The perpendicular from any point $q$ is then to that from $l_1$ as

$$S\,.\,qa : Sl_1 a, \tag{33}$$

for the perpendicular from $q$ is

$$Q_a = x' A_a + y' B_a + z' C_a + w' D_a\,,$$

$$\therefore \frac{Q_a}{A_a} = \frac{Sqa}{Sl_1 a}\,. \tag{34}$$

Hence any two perpendiculars as from $P$ and $Q$ are as

$$Spa : Sqa.$$

17. The volume of $(a_1, a_2, a_3, a_4)$ is

$$\pm\, \theta\Delta \,.\, Sa_1 Aa_2 a_3 a_4. \tag{35}$$

For the volume

$$\begin{array}{l} (a_1 - l_2 l_3 l_4\;) : (l_1 - l_2 l_3 l_4\;) = S\,.\,a_1 Al_2 l_3 l_4 \;: S\,.\,l_1 Al_2 l_3 l_4\;, \\ (a_2 - l_3 l_4 a_1) : (l_2 - l_3 l_4 a_1\;) = S\,.\,a_2 Al_3 l_4 a_1 \;: S\,.\,l_2 Al_3 l_4 a_1\;, \\ (a_3 - l_4 a_1 a_2) : (l_3 - l_4 a_1 a_2\,) = S\,.\,a_3 Al_4 a_1 a_2 \;: S\,.\,l_3 Al_4 a_1 a_2\;, \\ (a_4 - a_1 a_2 a_3) : (l_4 - a_1 a_2 a_3) = S\,.\,a_4 Aa_1 a_2 a_3 : S\,.\,l_4 Aa_1 a_2 a_3\,. \end{array}$$

Multiplying all together and representing the volume of the reference tetrahedron by $\Delta$,

$$(a_1 - a_2 a_3 a_4) = \pm\, \theta\Delta \,.\, Sa_1 Aa_2 a_3 a_4\,.$$

18. A dual interpretation is easily introduced at this point, for if in the present notation

$$S\,.\,ap = 0$$

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[NOTE.—$a_1$, $a_2$, $a_3$, $a_4$ must be unit points.]

denote a plane, we may assume an interpretation as follows: Let $a$ represent the plane whose perpendicular distances from $ABCD$ are as the elements of $a$, $a'$, $a''$, $a'''$, $a^{\text{IV}}$, then

$$S.ap = 0$$

will represent a point which is the isopole of this plane. The plane of the three points

$$S.pa_1 = 0, \quad S.pa_2 = 0, \quad S.pa_3 = 0$$

will then be $A.a_1a_2a_3$, etc.

19. If $a'\,b'\,c'\,d'$ represent the faces of the tetrahedron, we may define any conjugate of $P$ as

$$p = \Sigma x'l_1,$$
$$P_r = \Sigma \frac{r'}{x'} l_1 .$$

Many theorems of plane geometry are thus easily extended to solid geometry. I can only hint at them within the limits of this paper.

THE QUADRIC.

20. The general equation of the quadric is

$$S.p\Phi p = 0,$$

where $\Phi$ is a self-transverse linear quaternion operator. The line $p = a + x'b$ meets this surface at points corresponding to the two roots of the equation

$$x'^2 Sb\Phi b + 2x' Sa\Phi b + Sa\Phi a = 0. \tag{36}$$

The product of the roots is $\dfrac{Sa\Phi a}{Sb\Phi b}$; their sum, $-\dfrac{2Sa\Phi b}{Sb\Phi b}$. The roots are equal if

$$S.a\Phi a\, Sb\Phi b - S^2.a\Phi b = 0. \tag{37}$$

Hence if $b$ is a variable point satisfying this equation, we have the tangent cone with $a$ as vertex.

This cone touches the quadric at points satisfying the equation (37) and $Sb\Phi b = 0$, i. e. on the plane

$$S.b\Phi a = 0. \tag{38}$$

This is the polar plane of $a$. If $a$ is on the surface, it is the tangent plane.

The pole of $S.ap = 0$ is evidently therefore

$$\Phi^{-1}a. \tag{39}$$

Hence the center, or pole of the plane at infinity is

$$\Phi^{-1}g. \tag{40}$$

This is at infinity, and the surface is a paraboloid if

$$S.g\Phi^{-1}g = 0. \tag{41}$$

21. If one root of $\Phi$ is zero, then $m = 0$; and since

$$\Phi A.\Phi l_1 \Phi l_2 \Phi l_3 = 0,$$

in this case, $A.\Phi l_1 \Phi l_2 \Phi l_3$ must be a point on the surface. And since

$$S.p\Phi A.\Phi l_1 \Phi l_2 \Phi l_3 = 0,$$

for all values of $p$, i. e.

$$S.\Phi pA.\Phi l_1 \Phi l_2 \Phi l_3 = 0,$$

$A.\Phi l_1 \Phi l_2 \Phi l_3$ must lie on all polar planes, hence on all tangent planes. If $p$ is any other point on the surface,

$$A.\phi l_1 \phi l_2 \phi l_3 + x'p$$

evidently satisfies the equation of the surface. It must therefore be a cone whose vertex is

$$A.\Phi l_1 \Phi l_2 \Phi l_3.$$

In case two roots are zero, we have likewise $\Phi^2 \mathrm{X} l_1 = 0$, hence $\Phi \mathrm{X} l_1$ is a point on the surface. All tangent planes and all polar planes then go through two points, and the surface is therefore two planes.

In case three roots are zero, we find likewise that $\Phi^2 \Psi l_1$ is a point of the surface, which must then be one plane.

22. If the polar plane of $a$ pass through $b$, $Sa\Phi b = 0$, and this is the sole condition that the polar plane of $b$ pass through $a$.

23. The asymptotic cone is clearly

$$S.g\Phi^{-1}g\, Sp\Phi p - S^2 gp = 0, \tag{42}$$

or since $g$ is a scalar,

$$S.1\Phi^{-1}1.Sp\Phi p - S^2 p = 0. \tag{43}$$

24. The plane $Sap = 0$ touches the surface if it contain its own pole, hence the condition is

$$S.a\Phi^{-1}a = 0. \tag{44}$$

25. The polar line of $\quad A.pab = 0$
is easily found. For the polar plane of $a$ is $Sp\Phi a = 0$, and of $b$ is $Sp\Phi b = 0$. Hence the polar line is

$$A.rAa_x\Phi a\,\Phi b\,A.a_y\,\Phi a\Phi b = 0. \tag{45}$$

26. Referred to the invariant points of $\Phi$, $p_1, p_2, p_3, p_4$ we see at once that since

$$\Phi p_1 = p_1, \quad Sp_1p_2 = 0, \text{ etc.}$$

the invariant tetrahedron is self-polar with respect to the surface.

The usual theory is easily worked out from these first principles.

ILLINOIS COLLEGE, *August* 1, 1895.

# *On the Analytic Theory of Circular Functions.*

BY ALEXANDER S. CHESSIN.

## 1.—*Preliminaries.*

§1. It has become a usage with authors of treatises on the Theory of Functions to introduce the reader to the theory of doubly periodic functions by first treating simply periodic functions. Unfortunately the similarity between simply and doubly periodic functions ceases to exist when the behavior of the function at infinity comes to be investigated. Indeed, while in the case of doubly periodic and, in particular, of elliptic functions we treat these functions in a primitive parallelogram situated in the finite portion of the plane, we have to consider, in the case of simply periodic and, in particular, of circular functions, the *behavior of such functions at infinity, when the variable is restricted to remain within one of the primitive regions or bands into which the plane may be divided.* It was the neglect of this important point in the theory of simply periodic functions that led M. Forsyth to some erroneous conclusions in his excellent treatise on the Theory of Functions.* M. Méray, in his "Leçons nouvelles sur l'Analyse infinitésimale et ses applications géométriques,"† gives considerable attention to the point in question, which leads him to a classification of simply periodic functions into *polarized* and *non-polarized* functions.‡ However, the character and role of the *polar values*§ of a circular function have not yet been clearly set forth, and it is the object of the present paper to supply this deficiency.

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*See ch. X; also the review of this treatise by Mr. W. F. Osgood in the Bull. of the Amer. Math. Soc., 2d series, vol. I, no. 6.

†Vol. II, ch. VII.

‡Op. cit. p. 270. These terms correspond to the terms *circular* and *pseudo-circular* used in this paper.

§*Characteristic limits*, in the present paper.

§2. Let $f(z)$ be a uniform function admitting the single period $\omega$; then, by definition,

$$f(z+n\omega)=f(z), \qquad (1)$$
$$n\equiv\pm 1,\ \pm 2,\ \pm 3, \ldots .$$

This equation holding only for integral values of $n$, the period $\omega$ is *primitive*.

By means of a simple linear substitution of the variable the function $f(z)$ can be transformed into another uniform function admitting an arbitrarily assigned period $\omega'$. In fact, if we put

$$z=\frac{\omega}{\omega'}z';\ f(z)=f_1(z')$$

we shall have

$$f_1(z'+\omega')=f_1(z').$$

In particular, if $\omega$ be a complex number, we may put $\omega'=|\omega|$, so that the transformed function will admit a real period.

§3. It will be shown presently that the entire plane of $z$ may be divided into parallel bands in such a way that $f(z)$ takes all its values in each band. This proposition, the analogous of which is almost obvious in the case of doubly periodic functions, requires special considerations on account of the fact that the bands into which the plane of $z$ is divided extend to infinity.

We will agree to say that two points $z$ and $z'$ are congruent to each other with respect to $\omega$ if we have

$$z\equiv z' \pmod{\omega},$$

adopting this notation from the theory of numbers.

Let us draw any arbitrary line $L$ which does not cut itself and extends to infinity, and let a point $z$ describe this line; then the points congruent to $z$ will describe the system of lines

$$\ldots . L_{-2},\ L_{-1},\ L_0,\ L_1,\ L_2, \ldots .$$

parallel to each other and to the line $L$, the line $L_k$ being described by the point $z+k\omega$, and the entire plane of $z$ will be divided into parallel bands. Each band, i. e. the region contained between any two consecutive lines $L_{k-1}$ and $L_k$, will be called a *primitive region* of the function $f(z)$. For the sake of convenience we will say that the band formed by the lines $L_{k-1}$, $L_k$ is the $k^{\text{th}}$ primitive region. To each primitive region belongs one of the two lines forming it. We will assume that to the $k^{\text{th}}$ primitive region belongs the line $L_{k-1}$.

Let $z$ be any point within a fixed primitive region, for instance the $m^{\text{th}}$ region; all points congruent to $z$ will lie within other primitive regions, namely the point $z + k\omega$ will lie in the $(m+k)^{\text{th}}$ region.

If $z_1$ be an ordinary point of $f(z)$, then all points congruent to $z_1$ are ordinary points of the function. In fact, in the neighborhood of the point $z_1$ we can develop $f(z)$ in the form

$$f(z) = G(z - z_1), \tag{2}$$

where $G(z)$ denotes, as usually, an integral function. Let $z_2 \equiv z_1$ (mod. $\omega$) and $z'$ be any point in the neighborhood of the point $z_2$, i. e. let

$$z' = z_2 + \zeta; \qquad |\zeta| < \varepsilon. \tag{3}$$

Then by (1),

$$f(z') = f(z_2 + \zeta) = f(z_1 + \zeta),$$

and by (2) and (3),

$$f(z_1 + \zeta) = G(\zeta) = G(z' - z_2),$$

which proves the proposition.

If the point $z_1$ were an isolated singular point of $f(z)$—and this is the only kind of singularity we need to consider for our purposes—we would have instead of formula (2) the following one:

$$f(z) = G_1(z - z_1) + G_2\left(\frac{1}{z - z_1}\right). \tag{4}$$

Again, by (1),

$$f(z') = f(z_1 + \zeta),$$

and by (4) and (3),

$$f(z_1 + \zeta) = G_1(\zeta) + G_2\left(\frac{1}{\zeta}\right) = G_1(z' - z_2) + G_2\left(\frac{1}{z' - z_2}\right),$$

i. e. the function $f(z)$ has at the points $z_2$ exactly the same singularity as at the point $z_1$.

In a similar way it is readily shown that if $z_1$ be a vanishing point of the order $\lambda$ for $f(z)$, all points congruent to $z_1$ are vanishing points of the same order for this function.

It follows from the preceding remarks that the function $f(z)$ assumes in the finite portion of a primitive region all the values which it assumes in the finite portion of the entire plane, and that $f(z)$ is completely defined in the *finite portion* of the plane if it be defined in the *finite portion* of a primitive region. But it would be wrong to conclude without further investigation that $f(z)$ is completely defined in the *entire* plane if it be completely defined in a primitive region. It remains therefore to investigate the behavior of $f(z)$ at infinity, and to find out whether with regard to the infinity point also the function is characterized by its behavior at infinity *while z remains within a primitive region.*

§4. THEOREM I.—*The function $f(z)$ has an essentially singular point at infinity.*

This follows immediately from equation (1). In fact, if we put in it $n = \infty$ we obtain

$$f(\infty) = f(z),$$

i. e., the function assumes any arbitrarily assigned value at infinity, and the point $z = \infty$ is therefore an essentially singular point of the function.

THEOREM II.—*If the function $f(z)$ has an essentially singular point other than the point $z = \infty$, then it has an infinite number of essentially singular points.*

For if $z_1$ be an essentially singular point of $f(z)$ and $z_1 \neq \infty$, then all points congruent to $z_1$ are essentially singular points of the function (§3).

We will be concerned in this paper only with functions $f(z)$ having no other essentially singular point than the point $z = \infty$. Under this restriction $f(z)$ can have only isolated poles in the finite portion of a primitive region. In fact we know that whenever an infinite accumulation of poles takes place at a point, this point is an essentially singular point of the function. An infinite accumulation of poles may therefore take place only at infinity. This does not exclude the possibility of an infinite number of poles in a primitive region, provided there be only a finite number of them in the finite portion of such a region.

Let then $f(z)$ be a function as here defined. We can enunciate with regard to this function the following two propositions:

THEOREM III.—*If the function $f(z)$ has no pole in a primitive region, then it is an integral transcendental function, i. e.*

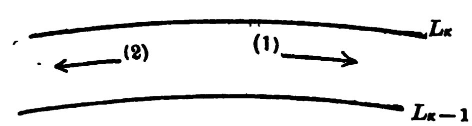
$$f(z) \equiv G(z).$$

THEOREM IV.—*If the function $f(z)$ has a finite number of poles in a primitive region or an infinite number of them, provided there be only a finite number of poles in the finite portion of the region, then $f(z)$ is the quotient of two integral functions, i. e.*

$$f(z) \equiv \frac{G_1(z)}{G_2(z)}.$$

These two theorems are an immediate consequence of Weierstrass's theory. With regard to the second theorem, it must be observed that one or the other of the functions $G_1(z)$, $G_2(z)$ may reduce to a constant, but if not constant they must be *transcendental* integral functions; because if $f(z)$ has a vanishing point or a pole in a primitive region, it will necessarily have an infinite number of vanishing points or poles in the whole plane (§3).

§5. Let $f(z)$ be again a function as defined in the preceding paragraph, and suppose that we restrict the variable to *remain within a fixed primitive region* of the function. If then we let $z$ tend to infinity and at the same time fix the direction of motion by one of the two arrows of the figure, the path of $z$ being




otherwise perfectly arbitrary, three possibilities arise: either $f(z)$ tends to a finite and determinate limit, or it tends to become infinite in a determinate way, or, finally, it becomes indeterminate. It will be shown later (§10) that if one of the enumerated possibilities takes place in one of the primitive regions, it will take place also in every other primitive region. It will be shown, moreover, that if one of the first two cases occurs for one of the directions indicated by the arrows, the same will be true for the other direction. Finally, it will be shown that this property is independent of the choice of the line $L$.

Functions $f(z)$, for which the first two of the enumerated cases take place, form a class by themselves and will be called *circular functions* in distinction from the functions for which the third of the enumerated cases takes place, and which will be called *pseudo-circular functions.*

THEOREM.—*A circular function can have only a finite number of poles in a primitive region.*

Suppose first that $\lim_{z=\infty} f(z)$ is finite and determinate when the variable remains in a primitive region and tends to infinity in the direction indicated by one of the two arrows of the figure. Let $N$ be this limit. Then we are able to assign a *neighborhood* of the point $z=\infty$ within the primitive region such that within this neighborhood $f(z)$ differs arbitrarily little from $N$. Now if $f(z)$ had an infinite number of poles in a primitive region, these would form an infinite accumulation at $z=\infty$ (§4), and it would be impossible to assign any neighborhood of $z=\infty$ in the primitive region such that in this neighborhood $f(z)$ differ arbitrarily little from $N$.

Suppose on the other hand that $f(z)$ tends to infinity in a determinate way when $z$ tends to infinity in the described manner. Then we are able to assign a neighborhood of the point $z=\infty$ within the primitive region such that within this neighborhood $f(z)$ have arbitrarily great but *finite* values. This again would be impossible if $f(z)$ had an infinite number of poles in a primitive region.

REMARK.—It will prove convenient to combine the two cases, namely, the case when $\lim\limits_{z=\infty} f(z)$ is finite and determinate, and the case when $f(z)$ tends to infinity in a determinate way, under the same category of *functions having a determinate limit* when $z$ tends to infinity in the manner described above. We will therefore agree that, unless explicitly expressed (*finite* and determinate), $\lim\limits_{z=\infty} f(z)$ may be infinite when we say that it has a determinate limit. It will, however, be remembered that this is only a convenient mode of expressing that $f(z)$ may *tend to infinity in a determinate way.*

## 2.—*Behavior of Circular Functions at Infinity.*

§6. Let us consider a circular function having a real period $\Omega$. We will choose for the line $L$ the axis of $y$, and we will assume that $\lim\limits_{z=\infty} f(z)$ has a determinate value when $z=x+iy$ tends to infinity while remaining in the first primitive region, the direction of motion being in this case indicated by the sign of $y$. We will denote the limits of $f(z)$ by $f_1$ and $f_2$ according as $y>0$ or $y<0$. These values may be infinite.* Let us next draw any arbitrary line $\Lambda$ extending to infinity. The angle formed by the direction of the motion of a point $z$ on this line and the positive axis of $y$, is a function of $z$, which we will denote by $\lambda(z)$. As the point $z$ moving on the line $\Lambda$ tends to infinity, the function $\lambda(z)$ may tend to a determinate limit, which we will denote by $\lambda(\infty)$, or it may become indeterminate.

THEOREM I.—*If $\lambda(z)$ tends to a determinate limit $\lambda(\infty)\neq\frac{\pi}{2}$, when $z$ tends to*

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* See Remark, §5.

*infinity along the line* $\Lambda$, *then* $\lim\limits_{z=\infty} f(z) = f_1$ *or* $f_2$, *according as* $\lambda(\infty) < \frac{\pi}{2}$ *or* $\lambda(\infty) > \frac{\pi}{2}$.

Let $z = x + iy$ be a point on the line $\Lambda$ and $z_0 = x_0 + iy_0$ its congruent in the first primitive region. Then

$$x \equiv x_0 \ (\text{mod. } \Omega)\,; \ y = y_0,$$
$$f(z) = f(x_0 + iy).$$

Let $z$ tend to infinity along the line $\Lambda$. Then $x_0$ will remain within the limits

$$0 < x_0 < \Omega,$$

and $x_0 + iy$ will remain within the first primitive region. Hence

$$\lim_{z=\infty} f(z) = \lim_{y=\pm\infty} f(x_0 + iy) = f_1 \text{ or } f_2,$$

according as $y = +\infty$ or $-\infty$, i. e. according as $\lambda(\infty) < \frac{\pi}{2}$ or $> \frac{\pi}{2}$, which proves the proposition.

REMARK.—*Theorem I still holds for* $\lambda(\infty) = \frac{\pi}{2}$ *provided* $x$ *and* $y$ *both tend to infinity with* $z$. The value of $\lim\limits_{z=\infty} f(z)$ in this case will be $f_1$ or $f_2$ according as $\lambda(z)$ remains from a certain place on $< \frac{\pi}{2}$ or $> \frac{\pi}{2}$ as $z$ tends to infinity along the line $\Lambda$.

THEOREM II.—*If* $\lambda(z)$ *tends to the determinate limit* $\lambda(\infty) = \frac{\pi}{2}$, *but* $y$ *does not become infinite with* $z$, *then* $\lim\limits_{z=\infty} f(z)$ *becomes indeterminate as* $z$ *tends to infinity along* $\Lambda$.

In this case the line $\Lambda$ has an asymptote parallel to the axis of $x$.* Let $a$ be its distance from this axis. Then it is clear that as $z$ tends to infinity along the line $\Lambda$, $x_0$ will again remain within the limits

$$0 \leqq x_0 < \Omega,$$

while $y$ will tend (or be equal) to the value $a$, so that

$$\lim_{z=\infty} f(z) = f(x_0 + ia),$$

* Or $\Lambda$ may simply become parallel to the axis of $x$.

where $x_0$ may have any value between 0 and $\Omega$. Therefore $\lim_{z=\infty} f(z)$ may have any value among those which $f(z)$ assumes on the line $y = a$ within the first primitive region, i. e. $\lim_{z=\infty} f(z)$ is indeterminate, which proves the theorem.

We can of course imagine that $z$ tends to infinity not along a continuous line, but by jumps from point to point. In this case it is no more possible to speak of an angle between the direction of the motion of the point $z$ and the positive axis of $y$. But it is always possible to draw a line $\Lambda$ such that the moving point will remain on this line, and as the preceding theorems are independent of the way in which the point $z$ may move on the line $\Lambda$, it is clear that they still remain true. The function $\lambda(z)$ will depend on the manner in which the line $\Lambda$ is drawn through the points passed by the moving point, but $\lambda(\infty)$ will be the same (if determinate) for all such lines $\Lambda$, because they will all be tangent to each other at the point $z = \infty$.

Heretofore we have assumed that $\lambda(\infty)$ had a determinate value. It remains to examine such paths $\Lambda$ for which $\lambda(\infty)$ is indeterminate. It is clear that all such paths may be divided into two classes. The paths of the first class are such that as $z$ tends to infinity along one of them $\lambda(z)$ may assume any value between 0 and $\pi$ *except the value* $\frac{\pi}{2}$. In this case $\lim_{z=\infty} f(z)$ will oscillate indefinitely between the values $f_1$ and $f_2$. The paths of the second class are characterized by the fact that as $z$ tends to infinity along one of them $\lambda(z)$ *may among other values assume the value* $\frac{\pi}{2}$.

THEOREM III.—*Whatever be the path along which $z$ tends to infinity, the value of $\lim_{z=\infty} f(z)$ will be either $f_1$ or $f_2$ or one of the values which $f(z)$ assumes in the finite portion of the first primitive region.*

In fact if $\lambda(\infty)$ has a determinate value, if this value be $\neq \frac{\pi}{2}$ we have seen that $\lim_{z=\infty} f(z) = f_1$ or $f_2$. If $\lambda(\infty) = \frac{\pi}{2}$, the value of $\lim_{z=\infty} f(z)$ is any one of the values of $f(z)$ on a line parallel to the axis of $x$ within the first primitive region, i. e. one of the values which $f(z)$ assumes in the finite portion of the first primitive region.

If $\lambda(\infty)$ is indeterminate but $\neq \frac{\pi}{2}$, we have seen that $\lim_{z=\infty} f(z)$ can have only the values $f_1$ and $f_2$, and if $\lambda(\infty)$ is entirely indeterminate, $\lim_{z=\infty} f(z)$ cannot only have the values $f_1$ and $f_2$ but also any value among those which $f(z)$ assumes on a line parallel to the axis of $x$ within the first primitive region, i. e. $\lim_{z=\infty} f(z)$ can have besides $f_1$ and $f_2$ any value of $f(z)$ in the finite portion of the first primitive region. Q. E. D.

§7. We have assumed at the beginning of the preceding paragraph that the circular function $f(z)$ with the real period $\Omega$ has a determinate limit when $z$ tends to infinity in any manner within the *first* primitive region, provided the sign of $y$ in $z = x + iy$ be ultimately the same,* and we have called this limit $f_1$ or $f_2$ according as the sign of $y$ is positive or negative. Let us now see how the function behaves at infinity in any other primitive region.

Suppose $z$ be restricted to remain within the $k^{\text{th}}$ primitive region, and let $z$ tend to infinity in such a way that $y$ have ultimately a fixed sign. Then it is clear that we can apply Theorem I of the preceding paragraph. In fact in this case $\lambda(\infty) = 0$ or $\pi$ according as $y$ becomes ultimately positive or negative. In the first case $\lim_{z=\infty} f(z) = f_1$, in the second $\lim_{z=\infty} f(z) = f_2$. Hence this

THEOREM I.—*If a function $f(z)$ with a real period tends to a determinate limit when $z$ tends to infinity while remaining in a fixed primitive region in such a way that $y$ have ultimately a determinate sign, then the function will tend to the same limit when $z$ tends to infinity while remaining in any other primitive region and the sign of $y$ being ultimately the same as in the first case.*

Now that we have seen that $\lim_{z=\infty} f(z)$ has the same value in every one of the primitive regions, it remains to find whether this property is independent of the manner in which the plane of $z$ has been divided into primitive regions. Let therefore $L$ be any line which does not cut itself and extends to infinity, and *which has no asymptote parallel to the axis of $x$ or which does not ultimately coincide with a line parallel to this axis.* If $z$ be restricted to remain within a fixed primi-

* This condition which here replaces the necessity of choosing one of the two arrows on the fig. of §5 is necessary, for otherwise $f(z)$ would oscillate indefinitely between the values $f_1$ and $f_2$ as $z$ tends to infinity.

tive region in this new division of the plane, it is clear that we can again apply Theorem I of the preceding §, and we shall have

$$\lim_{z=\infty} f(z) = f_1 \text{ or } f_2$$

according as $y$ becomes ultimately positive or negative. Hence this result:

*The preceding theorem remains true whatever be the manner in which the plane has been divided into primitive regions.*

It will be convenient for the future developments to fix a *positive direction of the line L*. In the present case (i. e. when the period of $f(z)$ is real) we will agree to take as positive that direction of the line $L$ which at infinity forms an acute angle with the positive axis of $y$. We will also agree to say that *z tends to infinity in the positive* (resp. negative) *direction, if the direction of its motion forms ultimately an acute* (resp. obtuse) *angle with the positive direction of the line L*. It is clear that if $z$ remains within a fixed primitive region, the limit of this angle is 0 (resp. $\pi$).

With these new definitions we can enunciate Theorem I in a more general form as follows:

THEOREM II.—*If a function f(z) with a real period tends to a determinate limit when z tends to infinity in a positive* (resp. negative) *direction while remaining in a fixed primitive region for a fixed division of the plane, then the function f(z) will tend to the same limit within every one of the primitive regions, as well for the given as for any other division of the plane, provided z tend to infinity always in the positive* (resp. negative) *direction.*

§8. The preceding discussion shows that with every circular function having a real* period are connected two fixed numbers $f_1$ and $f_2$† which are inasmuch characteristic for the function that they are independent of the manner in which the plane is divided into primitive regions, and that they are obtained as the limiting values of the function when the variable tends to infinity along any path having a determinate direction (as defined above), this limiting value being $f_1$ or $f_2$ according as the variable tends to infinity in the positive or negative

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* We shall see later that this restriction of the period is not necessary.

† These numbers may be equal to each other. One or both of them may also be infinite. (See Remark at the end of §5.)

direction. We will therefore call these numbers $f_1$ and $f_2$ *the characteristic limits at infinity*, or shorter, *the characteristic limits of the circular function* $f(z)$.

From Theorem III, §6, and Theorem II, §7, follows an important result, namely: *The function $f(z)$ whose period is real, assumes in a primitive region every assignable value.*

In fact, by Theorem III, §6, whatever be the path of $z$, the value of $\lim_{z=\infty} f(z)$ will be either $f_1$ or $f_2$ or any one of the values which $f(z)$ assumes in the finite portion of a fixed primitive region for a fixed division of the plane. By Theorem II, §7, the values $f_1$ and $f_2$ are the limits of $f(z)$ for $z=\infty$ also in any other primitive region and independently of the manner in which the plane is divided into parallel bands. We know on the other hand that all the values assumed by $f(z)$ in the finite portion of the entire plane are also assumed by the function in the finite portion of any one of its primitive regions, independently of the division of the plane (§3). Hence all the values of $\lim_{z=\infty} f(z)$ are those which $f(z)$ assumes in a primitive region, the numbers $f_1$ and $f_2$, i. e. the values of $f(z)$ at infinity in the primitive region included. But we know that at the point $z=\infty$, which is an essentially singular point of $f(z)$, the function assumes every assignable value; hence $f(z)$ assumes every assignable value in a primitive region. Q. E. D.

If we recall the manner in which the several values of $\lim_{z=\infty} f(z)$ are obtained, we will notice that all the values except possibly $f_1$ and $f_2$ can be obtained by making $z$ tend to infinity along lines having asymptotes parallel to the axis of $x$, and the distances of these asymptotes from the axis of $x$ assuming every possible negative or positive value. The simplest manner of drawing these lines is to take them parallel to the axis of $x$. We can therefore say that

*Every assignable value except possibly the characteristic limits of a circular function having a real period, can be obtained for $\lim_{z=\infty} f(z)$ by making $z$ tend to infinity along a line parallel to the axis of $x$.*

We shall see later (§10), that in the general case we only need to substitute for the axis of $x$ the line forming with it the angle $\arg(\omega)$, $\omega$ being the period of the function.

§9. When we defined the function $f(z)$ in §6 we assumed that $\lim_{z=\infty} f(z)$ had a determinate value for either of the two directions in which $z$ may tend to infinity in a primitive region. The question naturally presents itself: is it necessary to assume that $\lim_{z=\infty} f(z)$ is determinate for either direction, or does it suffice to obtain a determinate limit for only one direction to assure the determinateness of the limit for the other?

THEOREM.—*If a function $f(z)$ having a real period tends to a determinate limit when $z$ tends to infinity in the positive direction while remaining in a primitive region, then the function will also tend to a determinate limit when $z$ tends to infinity in the negative direction.*

In fact, suppose that the function tends to a determinate limit when $z$ tends to infinity along a given fixed path $\Lambda$ in the positive direction. To this path we can correlate another $\Lambda'$ symmetrical with respect to the axis of $x$. Two correlated points on $\Lambda$ and $\Lambda'$ are then $x+iy$ and $x-iy$. Now, we can always give to the function $f(z)$ the form

$$f(z) = U(z) + iV(z), \tag{5}$$

where $U(z)$ and $V(z)$ are *real* functions of the variable $z$ as long as $z$ is *real*. At the same time we may put

$$\left.\begin{aligned} U(z) &= U_1(x, y) + iU_2(x, y), \\ V(z) &= V_1(x, y) + iV_2(x, y), \end{aligned}\right\} \tag{6}$$

so that if

$$f(z) = [U_1(x, y) - V_2(x, y)] + i[U_2(x, y) + V_1(x, y)] \tag{7}$$

gives the value of the function for a point $(x, y)$ of $\Lambda$, the value of the function for the correlated point on $\Lambda'$ will be

$$f(z) = [U_1(x, y) + V_2(x, y)] + i[-U_2(x, y) + V_1(x, y)]. \tag{8}$$

A simple glance at the expressions (7) and (8) shows that if one of them tends to a determinate limit when $z$ tends to infinity, the other will also tend to a determinate limit.

Now if along $\Lambda$ $z$ tends to infinity in the positive direction, it is clear that along $\Lambda'$ it tends to infinity in the negative direction. But as $\lim_{z=\infty} f(z)$ is determinate when $z$ tends to infinity in the positive direction while remaining in a primitive region, $\lim_{z=\infty} f(z)$ will have a determinate value whatever be the path

$\Lambda$ along which $z$ tends to infinity* in the positive direction (Theor. II, §7). Hence $\lim_{z=\infty} f(z)$ will have a determinate value whatever be the path $\Lambda'$ along which $z$ tends to infinity* in the negative direction. Q. E. D.

REMARK.—It is obvious that the functions $U(z)$ and $V(z)$ are simply periodic and have the same period as $f(z)$. Moreover $\lim_{z=\infty} U(z)$ and $\lim_{z=\infty} V(z)$ have a determinate value when $z$ tends to infinity in the manner described above. Hence $U(z)$ and $V(z)$ are *circular* functions like $f(z)$.

COROLLARY I.—*We can determine the characteristic limits of a circular function $f(z) = U(z) + iV(z)$ having a real period if we know one of the characteristic limits for each of the functions $U(z)$ and $V(z)$, provided these be not both infinite.*

In fact suppose, to fix the ideas, that we know the characteristic limit of $U(z)$ when $z$ tends to infinity in the positive direction, and let it be $u' + iu''$; also the characteristic limit of $V(z)$ for instance when $z$ tends to infinity in the negative direction, and let it be $v' + iv''$. Then, by formulas (6),

$$u' = \lim_{z=\infty} U_1;\quad u'' = \lim_{z=\infty} U_2,$$
$$v' = \lim_{z=\infty} V_1;\quad v'' = -\lim_{z=\infty} V_2,$$

and therefore by (7) and (8),

$$\left.\begin{aligned} f_1 &= u' + v'' + i(\phantom{-}u'' + v'),\\ f_2 &= u' - v'' + i(-u'' + v').\end{aligned}\right\} \tag{9}$$

If only one of the numbers $u' + iu''$ or $v' + iv''$ be infinite, the corollary still holds. In fact then $f_1 = f_2 = \infty$, and the proposition is thus proved.

REMARK.—If both $u' + iu''$ and $v' + iv''$ were infinite, it would be necessary to go back to the expressions (7) and (8) and then make $z$ tend to infinity in order to obtain the limits $f_1$ and $f_2$.

COROLLARY II.—*If the characteristic limits $f_1$ and $f_2$ of a circular function having a real period are finite, then the condition necessary and sufficient in order that $f_1 = f_2$ is that the characteristic limits of the functions $U(z)$ and $V(z)$ be real.*

This follows immediately from formulas (9), which give the condition $v'' = u'' = 0$.

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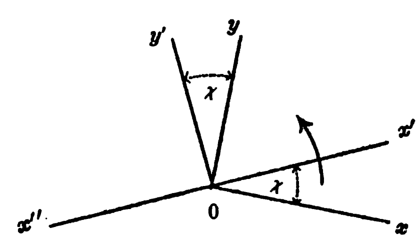
* Always excluding such lines as tend to become parallel to the axis of $x$ at infinity.

The discussion of this paragraph shows that the propositions advanced when defining circular functions in §5 are true at least in the case of a real period. It will be shown presently that they are always true.

§10. Heretofore it has been assumed that the period of the circular function was real (§6). But we can at once extend our results to the general case when the period $\omega$ of $f(z)$ is any complex number if we make use of the linear substitution mentioned in §2. In fact, we put again

$$\Omega = |\omega|; \quad z = \frac{\omega}{\Omega} z'; \quad f(z) = F(z'),$$

and then $F(z')$ is a circular function admitting the real period $\Omega$. We may therefore apply the results of §§6–9 to the function $F(z')$. It will be noticed that the positive axis of $x'$ forms the angle $\chi \equiv \arg(\omega)$ with the positive axis of $x$. Positive angles will be counted in the direction indicated by the arrow on the figure. The line $x'x''$ will be called *the line of periodicity of the function*




$f(z)$, and the direction $0x'$ the *positive direction of the line of periodicity.* We will also agree to call the direction $0y'$ the *normal to the line of periodicity.*

With these definitions it becomes clear that we only need to substitute the line of periodicity for the axis of $x$ in the results of §§6–9 to make them valid in the general case, i. e. when the period of $f(z)$ is any complex number.

The function $\lambda(z)$ used in §§6 and 7 will now be the angle between the direction of the motion of $z$ when it tends to infinity along a line $\Lambda$, and the normal to the line of periodicity. The enunciation of the three theorems of §6 need not be changed.

Coming to §7 it is necessary first to agree as to the terms "positive direction of $L$" and "$z$ tends to infinity in the positive direction." The positive direction of $L$ will be that which at infinity forms an acute angle with the normal to the line of periodicity; and we will say that $z$ tends to infinity in the positive (resp.

negative) direction if the direction of its motion ultimately forms an acute (resp. obtuse) angle with the positive direction of the line $L$. It will be assumed that the line $L$ has no asymptotes parallel to the line of periodicity (or that it does not ultimately coincide with a line parallel to it).

With these more general definitions we can at once extend Theorem II §7 to functions $f(z)$ with a complex period $\omega$. As to Theorem I, it is only another form of the same proposition. The results of §8, namely, the *existence of two characteristic limits*, and the proposition that *a circular function assumes every assignable value in a primitive region*, are immediately extended to any circular function. As to the last proposition of §8 it will now be formulated as follows:

*Every assignable value except possibly the characteristic limits of a circular function* $f(z)$ *can be obtained for* $\lim_{z=\infty} f(z)$ *by making* $z$ *tend to infinity along a line parallel to the line of periodicity.*

The Theorem of §9 obviously holds for any circular function; to formulate it for the general case again we only need to substitute the line of periodicity for the axis of $x$. As to the corollaries to this theorem, they evidently hold only for circular functions with a real period. However they may still be used with advantage. In fact the characteristic limits of the functions $f(z)$ and $F(z')$ being identical, we may by applying these corollaries to the function $F(z')$, either determine the characteristic limits of $f(z)$ (Cor. I) or decide whether these limits are equal (Cor. II).

It follows from the preceding discussion that the propositions advanced when defining circular functions in §5 are true. This definition may therefore be formulated as follows:

DEFINITION.—*A circular function is a uniform simply periodic function with no other essentially singular point than the point* $z = \infty$ *and such that* $\lim_{z=\infty} f(z)$ *has a determinate value when* $z$ *tends to infinity in a fixed direction while remaining in a primitive region.*

A function so defined tends to a determinate limit when $z$ tends to infinity along any fixed path in a fixed direction; moreover it possesses two characteristic limits which may be equal.

It also follows from the preceding discussion that a pseudo-circular function cannot tend to a determinate limit when $z$ tends to infinity while remaining within a primitive region, *whatever be the division of the plane*, i. e. whatever be the line $L$. But a pseudo-circular function may tend to a determinate limit when $z$ tends to infinity along some particular paths within a primitive region. For example, the function $e^{\sin z}$ tends to zero when $z$ tends to infinity along parallels to the axis of $y$ at the distances $(4n+3)\frac{\pi}{2}$ where $n \equiv 0, \pm 1, \pm 2, \ldots$

As another example of a pseudo-circular function may be mentioned Hermite's function $Z(z)$; its derivative is an elliptic function. The function $Z(z)$ tends to no determinate value when $z$ tends to infinity remaining in a primitive region, whatever be the path chosen.

It would be interesting to investigate more closely the character of this distinction of pseudo-circular functions, but this paper is restricted to the study of circular functions alone.

§11. THEOREM.—*A circular function which is finite everywhere in a primitive region is a constant.*

In fact suppose that the function $f(z)$ is finite in the finite portion of a primitive region and that its characteristic limits $f_1$ and $f_2$ are also finite. All the values of $f(z)$ at infinity except perhaps the values $f_1$ and $f_2$ can be obtained by making $z$ tend to infinity along lines parallel to the line of periodicity (§10). But $f(z)$ being finite everywhere in a primitive region, the values of $f(z)$ at infinity so obtained are necessarily also finite; hence all the values of $f(z)$ at infinity are finite, i. e. the function is holomorphic in the entire plane of $z$, and therefore it reduces to a constant.

COROLLARY I.—*A circular function becomes infinite at least once within a primitive region.*

The function may have one or more poles in a primitive region or one or both of its characteristic limits may be infinite. If $f(z)$ has no pole in a primitive region, then at least one of its characteristic limits is infinite. If both its characteristic limits are finite, $f(z)$ has at least one pole in a primitive region.

COROLLARY II.—*A circular function $f(z)$ assumes every assignable value at least once in a primitive region.*

This follows at once from the preceding Corollary if we consider the circular function

$$\frac{1}{f(z)-A},$$

where $A$ is any arbitrarily assigned number.

§12. *Picard's theorem.*

If we recollect how the different values of $f(z)$ at infinity may be obtained, we see that there will be an infinite number of points in the neighborhood of the point $z = \infty$, at which the function assumes any arbitrarily assigned value except perhaps the values of the characteristic limits: $f_1$ and $f_2$. In fact all other values may be obtained by making $z$ tend to infinity along lines parallel to the line of periodicity (§10), and each one of these values will repeat itself an infinite number of times as $z$ tends to infinity. We thus obtain Picard's theorem* for the particular case of circular functions in the following form:

*In the neighborhood of the essentially singular point of a circular function there is an infinite number of points at which the function assumes any arbitrarily assigned value except possibly one or two values.*

We may add that—

*these exceptional values are the characteristic limits of the circular function, and that there will be one or two exceptional values according as the characteristic limits are or are not equal.*

We may furthermore add that—

*all values of* $\lim\limits_{z=\infty} f(z)$ *other than* $f_1$ *and* $f_2$ *may be obtained by making* $z$ *tend to infinity along lines parallel to the line of periodicity, and that the values* $f_1$ *and* $f_2$ *will or will not be exceptional according as the function does not or does assume these values in the finite portion of a primitive region; and finally, that either the function assumes the values* $f_1$ *and* $f_2$ *at an infinite number of points in the neighborhood of the point* $z = \infty$, *or it assumes them only at the point* $z = \infty$ *itself.*

§13. We are able now to answer the question formulated at the end of §3, namely, in the affirmative sense: with regard to the point $z = \infty$ as with regard to any other point a circular function is characterized by its behavior in a primitive region, and we can enunciate the following fundamental

THEOREM.—*A circular function is defined in the entire plane if it is defined in a primitive region.*

This theorem is independent of the manner in which the plane is divided into parallel bands.

We may therefore restrict our investigation to the study of circular functions in a primitive region.

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* "Mémoire sur les Fonctions entières," An. de l'École Normale, 1880.

3.—*Study of Circular Functions within a Primitive Region.*

§14. THEOREM I.—*If two circular functions* $f(z)$ and $\phi(z)$ *admitting the same period* $\omega$ *and whose characteristic limits at infinity are finite and* $\neq 0$ *have the same vanishing points and poles with the same respective orders of multiplicity in a primitive region, then*

$$f(z) = C\phi(z).$$

In fact the function $\frac{f(z)}{\phi(z)}$ is a circular function which becomes infinite nowhere in a primitive region, and therefore reduces to a constant (§11).

THEOREM II (generalization of the preceding theorem).—*If two circular functions* $f(z)$ *and* $\phi(z)$ *admitting the same period* $\omega$ *have the same vanishing points and poles with the same respective orders of multiplicity in a primitive region, and if at the same time the characteristic limits at infinity of the circular function* $\frac{f(z)}{\phi(z)}$ *are finite and* $\neq 0$, *then* $f(z) = C\phi(z)$.

In fact $\frac{f(z)}{\phi(z)}$ is again finite everywhere in a primitive region and therefore reduces to a constant.

THEOREM III.—*If two circular functions admitting the same period and whose characteristic limits at infinity are finite, have the same poles with the same respective orders of multiplicity in a primitive region; moreover, if* $\beta_i$ *being any one of these poles and* $\mu_i$ *its order of multiplicity, the coefficients of the different powers of* $\frac{1}{z-\beta_i}$ *in the developments of the two functions in the neighborhood of the point* $\beta_i$ *be respectively the same, then*

$$f(z) = \phi(z) + C.$$

In fact the circular function $f(z) - \phi(z)$ is finite throughout a primitive region, and therefore reduces to a constant.

This theorem may also be generalized as follows:

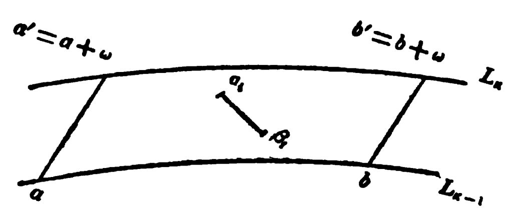
THEOREM IV.—*If two circular functions admitting the same period have the same poles with the same respective orders of multiplicity in a primitive region; moreover, if* $\beta_i$ *being any one of these poles and* $\mu_i$ *its order of multiplicity, the coefficients of the different powers of* $\frac{1}{z-\beta_i}$ *in the developments of the two functions in the* *neighborhood of the point $\beta_i$ be respectively the same; and if at the same time the characteristic limits of the circular function $f(z) - \phi(z)$ be finite, then*

$$f(z) = \phi(z) + C.$$

The demonstration is again based on the fact that the circular function $f(z) - \phi(z)$ is finite throughout a primitive region.

§15. THEOREM.—*A circular function $f(z)$ takes every assignable value except possibly $f_1$ and $f_2$ the same number of times in a primitive region. In particular it vanishes as many times as it becomes infinite in a primitive region, except possibly when $f_1$ or $f_2$ are 0 or $\infty$.*

We will assume at first that the characteristic limits of the function are finite and $\neq 0$. The line $L$ being drawn arbitrarily, we can assume that it passes through no pole or vanishing point of the function. Then none of the lines $L_\kappa$ will pass through a pole or vanishing point of the function.




On the other hand, as $f_1$ and $f_2$ are finite and $\neq 0$, we may choose two points $a$ and $b$ on the line $L_{\kappa-1}$ in such a manner that all those points of the $\kappa^{\text{th}}$ primitive region which are poles $(\beta_i)$ or vanishing points $(a_i)$ of $f(z)$ will lie inside of the curvilinear parallelogram $(abb'a')$.

Consider the integral

$$\frac{1}{2\pi i}\int \text{dlg}\, f(z) \tag{10}$$

taken along the boundary of this parallelogram. The path of integration may be broken up into four parts: 1) from $a$ to $b$; 2) from $b$ to $b'$; 3) from $b'$ to $a'$; 4) from $a'$ to $a$. It is obvious that the first and the third of these integrals cancel each other, since the function has the same value at congruent points. As to the remaining two integrals, each one vanishes separately. In fact

$$\int_b^{b+\omega} \text{dlg}\, f(z) = \lg f(b+\omega) - \lg f(b) = 0,$$

because we know that this integral is single-valued as long as the path of integration does not cross any of the lines $\alpha_i\beta_i$ connecting the poles and vanishing points of $f(z)$. For a similar reason

$$\int_{a+\omega}^{a} \operatorname{dlg} f(z) = 0,$$

and therefore the integral (10) vanishes. But we know that this integral is equal to the difference between the number of times $n$ that $f(z)$ vanishes and the number of times $m$ that it becomes infinite inside of the curvilinear parallelogram $(abb'a')$. Hence $m = n$, and as $f(z)$ neither vanishes nor becomes infinite in the remaining portion of the primitive region, the theorem is partly proved. To complete the proof, consider the circular function

$$\phi(z) = f(z) - N,$$

where $N$ is any arbitrarily assigned number other than $f_1$ and $f_2$. Then we can apply to the function $\phi(z)$ the proposition just proved, namely, $\phi(z)$ vanishes as many times as it becomes infinite in a primitive region. But $\phi(z)$ and $f(z)$ become infinite the same number of times $(n)$ in a primitive region. Hence $f(z)$ assumes the value $N$ also exactly $n$ times in a primitive region. Q. E. D.

Suppose now that one or both of the characteristic limits of $f(z)$ may be 0 or $\infty$. Consider in this case the circular function

$$F(z) \equiv A + \frac{1}{f(z) + B},$$

where $A$ and $B$ are any fixed numbers $\neq 0$ satisfying the condition $AB + 1 \neq 0$. Then the characteristic limits $F_1$ and $F_2$ of the function $F(z)$ are finite and $\neq 0$. It has been just proved that $F(z)$ assumes any arbitrarily assigned value except possibly $F_1$ and $F_2$ the same number of times in a primitive region. Hence the circular function $f(z)$ assumes any arbitrarily assigned value, except possibly $f_1$ and $f_2$ the same number of times in a primitive region. Q. E. D.

The number of times that a circular function $f(z)$ assumes in a primitive region any arbitrarily assigned value except possibly $f_1$ and $f_2$ is therefore a constant. This constant will be called *the order of the circular function.*

§16. Let $\theta(z)$ be a circular function of the first order admitting the period $\omega$, and let this function be finite and $\neq 0$ in the finite portion of a primitive region. Then one of its characteristic limits is $\infty$ and the other is $0$. The existence of such functions can be established *a priori*. In fact such is, for example, the function

$$1 + \frac{z}{1} + \frac{z^2}{2!} + \ldots . + \frac{z^n}{n!} + \ldots . , \tag{11}$$

which is denoted by the symbol $e^z$ and whose period is $2\pi i$.* The existence of this function will be proved also *a posteriori* in §29.

Let $f(z)$ be any other circular function of the first order admitting the period $\omega'$. We can always choose two numbers $A$ and $B$ such that the characteristic limits of the circular function of the first order

$$F(z) = A + \frac{1}{f(z) + B} \tag{12}$$

be finite and $\neq 0$. In fact, we only need to satisfy the conditions

$$B \neq \begin{cases} -f_1 \\ -f_2 \end{cases}; \quad \frac{AB+1}{A} \neq \begin{cases} -f_1 \\ -f_2 \end{cases}.$$

Then the function $F(z)$ has a pole and a vanishing point both of the first order in a primitive region. Let $\alpha$ be the vanishing point and $\beta$ the pole in the same primitive region. We can by means of the function $\theta(z)$ construct a circular function of the first order having the period $\omega'$, whose characteristic limits are finite and whose vanishing point and pole in a primitive region are respectively $\alpha$ and $\beta$. In fact, such is the function

$$\Theta(z) \equiv \frac{\theta\left(\frac{\omega}{\omega'} z\right) - \theta\left(\frac{\omega}{\omega'} \alpha\right)}{\theta\left(\frac{\omega}{\omega'} z\right) - \theta\left(\frac{\omega}{\omega'} \beta\right)}. \tag{13}$$

Hence, by Theorem I, §14,

$$F(z) = C\Theta(z),$$

and therefore

$$f(z) = \frac{m\theta\left(\frac{\omega}{\omega'} z\right) + n}{p\theta\left(\frac{\omega}{\omega'} z\right) + q}, \tag{14}$$

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* That all the properties of the exponential function can be derived directly from its definition by the infinite series (11) is shown in most of the modern text-books on the elements of the Theory of Functions. See, for example, Méray, op. cit. II, p. 207.

in which we have put

$$m = 1 + AB - BC,$$
$$n = BC\theta\left(\frac{\omega}{\omega'}\alpha\right) - (1 + AB)\,\theta\left(\frac{\omega}{\omega'}\beta\right),$$
$$p = C - A,$$
$$q = A\theta\left(\frac{\omega}{\omega'}\beta\right) - C\theta\left(\frac{\omega}{\omega'}\alpha\right).$$

Formula (14) expresses the following

THEOREM I.—*Every circular function of the first order can be expressed as a linear fractional function of a given circular function of the first order, which is finite and* $\neq 0$ *in the finite portion of a primitive region.*

More generally we have the

THEOREM II.—*Every circular function of the first order can be expressed as a linear fractional function of any other circular function of the first order.*

In fact, if $\phi(z)$ were a given circular function of the first order with the period $\omega''$, we would have by formula (14)

$$\phi\left(\frac{\omega''}{\omega'}z\right) = \frac{m'\theta\left(\frac{\omega}{\omega'}z\right) + n'}{p'\theta\left(\frac{\omega}{\omega'}z\right) + q'},$$

and the elimination of $\theta\left(\frac{\omega}{\omega'}z\right)$ between the last equation and (14) gives an expression of the form

$$f(z) = \frac{m''\phi\left(\frac{\omega''}{\omega'}z\right) + n''}{p''\phi\left(\frac{\omega''}{\omega'}z\right) + q''},$$

which proves the theorem.

If we put $\theta(z) = e^{z}$ formula (14) takes the particular form

$$f(z) = \frac{me^{\frac{2\pi i}{\omega'}z} + n}{pe^{\frac{2\pi i}{\omega'}z} + q}. \qquad (15)$$

COROLLARY.—*All circular functions of the first order which remain finite and* $\neq 0$ *in the finite portion of a primitive region have the form*

$$Ce^{kz}.$$

In fact, let $f(z)$ be a circular function of the first order which remains finite in the finite portion of a primitive region. Then its characteristic limits are 0 and $\infty$. Formula (15) shows that in this case either $m=q=0$ or $n=p=0$, so that $f(z)$ has one of the two forms

$$\frac{m}{q}e^{\frac{2\pi i}{\omega'}z} \text{ or } \frac{n}{p}e^{-\frac{2\pi i}{\omega'}z},$$

which proves the proposition.

§17. Formula (14) shows that

*The characteristic limits at infinity of a circular function of the first order are different from any of the values of the function in the finite portion of a primitive region.* It also shows that *the two characteristic limits of such a function are different.*

Picard's theorem (§12) in this case reads as follows: in the neighborhood of the essentially singular point of a circular function of the first order $f(z)$ there is an infinite number of points at which the function assumes any arbitrarily assigned value except the two values $f_1$ and $f_2$, which are different and which the function assumes only at the point $z=\infty$ itself.

§18. We will use the exponential function to derive a criterion for the manner in which a circular function tends to its characteristic limits. We will denote the function $e^{\frac{2\pi i}{\omega}z}$ by the symbol $u_\omega$, and we will say that the positive direction of the line $L$ is that for which $\lim\limits_{z=\infty} u_\omega = 0$. This is in accordance with the former definition of the positive direction of the line $L$ (§10).

Let now $f(z)$ be a circular function with the period $\omega$, whose characteristic limit in the *negative* direction is infinite. If a positive integer $n$ can be found such that

$$\lim_{z=\infty}\left[\frac{f(z)}{u_\omega^n}\right] = N\begin{cases}\neq 0\\ \neq \infty\end{cases}, \tag{16}$$

when $z$ tends to infinity in the negative direction, we will say that *the characteristic limit of the function $f(z)$ is an exponential infinity of the $n^{th}$ order.* If the characteristic limit of $f(z)$ in the *positive* direction is infinite and a positive integer $n$ can be found such that

$$\lim_{z=\infty}\left[u_\omega^n f(z)\right] = N\begin{cases}\neq 0\\ \neq \infty\end{cases}, \tag{16$'$}$$

when $z$ tends to infinity in the positive direction, we will say again that *the characteristic limit of $f(z)$ is an exponential infinity of the $n^{th}$ order.*

If the characteristic limit of $f(z)$ is zero, we will say that it is *an exponential zero of the $n^{th}$ order* if a positive integer $n$ can be found such that

$$\lim_{z=\infty} [u_\infty^n f(z)] = N \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \tag{17}$$

or that

$$\lim_{z=\infty} \left[\frac{f(z)}{u_\infty^n}\right] = N \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \tag{17'}$$

according as we deal with a characteristic limit of $f(z)$ in the negative or in the positive direction.

More generally, if a characteristic limit of $f(z)$ is the number $A$, we will say that *the function $f(z)$ assumes at this limit the value $A$ exponentially $n$ times if a positive integer $n$ can be found such that*

$$\lim_{z=\infty} [u_\infty^n (f(z) - A)] = N \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \tag{18}$$

*or that*

$$\lim_{z=\infty} \left[\frac{f(z) - A}{u_\infty^n}\right] = N \begin{cases} \neq 0 \\ \neq \infty \end{cases}, \tag{18'}$$

*according as we deal with a characteristic limit of $f(z)$ in the negative or in the positive direction.*

These definitions are in perfect harmony with the ordinary definitions of vanishing points and poles. In fact, if the function $f(z)$ has a vanishing point $\alpha$ of the order $\lambda$ or a pole $\beta$ of the order $\mu$, it is clear that in the first case

$$\lim_{z=\alpha} \frac{f(z)}{[e^z - e^\alpha]^\lambda} = N \begin{cases} \neq 0 \\ \neq \infty \end{cases},$$

and in the second case

$$\lim_{z=\beta} [(e^z - e^\beta)^\mu f(z)] = N \begin{cases} \neq 0 \\ \neq \infty \end{cases}.$$

These ordinary definitions fail at the point $z = \infty$. But they may be extended to this point, provided the variable be restricted to approach it in a certain manner only, namely, along any path which does not tend to become parallel to the line of periodicity at infinity.

This restriction being imposed only on the point $z = \infty$, it is clear that we can omit the word "exponentially" when we speak of $f(z)$ assuming a certain value at one of its characteristic limits. Indeed, the specification of the place at which $f(z)$ assumes this value, namely, the words "at the characteristic limits," implies that the variable is restricted to remain within a primitive region. Likewise we may omit the word "exponential" when speaking of the zeros and the infinities of the function $f(z)$ if we specify how many of them are situated at the characteristic limits. For instance, if we say that $f(z)$ has $n$ infinities in a primitive region of which $p$ are at infinity, then these last $p$ infinities are exponential infinities of $f(z)$ in a primitive region while the other $n-p$ are poles.

§19. It follows from the new definitions that *a circular function of the first order assumes the values of its characteristic limits only once in a primitive region.* In fact, if $\omega$ be the period of such a function, we have by formula (15)

$$f(z)=\frac{mu_\omega+n}{pu_\omega+q}.$$

Let $f_1$ and $f_2$ be the characteristic limits of $f(z)$ in the positive $(u_\omega=0)$ and respectively negative $(u_\omega=\infty)$ direction. Then

$$f_1=\frac{n}{q};\quad f_2=\frac{m}{p}.$$

We will first assume that $p$ and $q$ are not equal to zero. Then

$$\lim_{\substack{z=\infty\\ u_\omega=\infty}} u_\omega\,[f(z)-f_2]=\frac{np-mq}{p^2}\left\{\begin{array}{l}\neq 0\\ \neq\infty\end{array}\right.,$$

$$\lim_{\substack{z=\infty\\ u_\omega=0}}\left[\frac{f(z)-f_1}{u_\omega}\right]=\frac{mq-np}{q^2}\left\{\begin{array}{l}\neq 0\\ \neq\infty\end{array}\right.,$$

which proves the proposition. If, on the other hand, $q=0$ or $p=0$, the function $f(z)$ takes one or the other of the two forms: $C+\frac{C'}{u_\omega}$ or $Cu_\omega+C'$. In either case the proposition becomes obvious.

If we take into account the remark made in §17, namely, that a circular function of the first order assumes the values of its characteristic limits within a primitive region only at these limits, then the proposition of this paragraph may be enunciated in the form of the following

THEOREM.—*A circular function of the first order assumes every assignable value only once in a primitive region.*

§20. Let $f(z)$ be a circular function admitting the period $\omega$, and let in a primitive region $\alpha_1, \alpha_2, \ldots, \alpha_p$ be its vanishing points of the respective orders of multiplicity $\lambda_1, \lambda_2, \ldots, \lambda_p$; $\beta_1, \beta_2, \ldots, \beta_q$ its poles of the respective orders $\mu_1, \mu_2, \ldots, \mu_q$; moreover, let the characteristic limits of $f(z)$ be finite and $\neq 0$. Then (§15) $\sum_1^p \lambda_\kappa = \sum_1^q \mu_\kappa = n$, $n$ being the order of the circular function.

Let $\theta(z)$ be a circular function of the first order admitting the period $\omega$ and which is finite and $\neq 0$ in the finite portion of a primitive region. Then the function

$$\Theta(z) = \frac{[\theta(z) - \theta(\alpha_1)]^{\lambda_1}[\theta(z) - \theta(\alpha_2)]^{\lambda_2} \ldots [\theta(z) - \theta(\alpha_p)]^{\lambda_p}}{[\theta(z) - \theta(\beta_1)]^{\mu_1}[\theta(z) - \theta(\beta_2)]^{\mu_2} \ldots [\theta(z) - \theta(\beta_q)]^{\mu_q}} \tag{19}$$

is a circular function with the period $\omega$, whose vanishing points and poles are the same and of the same respective orders of multiplicity as those of $f(z)$. Moreover, the characteristic limits of $\Theta(z)$ are finite and $\neq 0$,

$$\Theta_1 = 1,$$

$$\Theta_2 = \frac{[-\theta(\alpha_1)]^{\lambda_1}[-\theta(\alpha_2)]^{\lambda_2} \ldots [-\theta(\alpha_p)]^{\lambda_p}}{[-\theta(\beta_1)]^{\mu_1}[-\theta(\beta_2)]^{\mu_2} \ldots [-\theta(\beta_q)]^{\mu_q}}.$$

Hence (Theorem I, §14)

$$f(z) = C\Theta(z), \tag{19'}$$

*i. e. the function $f(z)$ is a rational function of $\theta(z)$.* The degree of the numerator in (19) is equal to that of the denominator and to $n$, i. e. to the order of the function $f(z)$. This fact is due to the condition that the characteristic limits of $f(z)$ be finite and $\neq 0$.

Let us now consider any circular function of the $n^{\text{th}}$ order. We can always find two numbers $A$ and $B$ such that the characteristic limits of the circular function of the $n^{\text{th}}$ order

$$F(z) = A + \frac{1}{f(z) + B}$$

be finite and $\neq 0$ (§16). Then, as has been just proved, $F(z)$ is a rational function of the circular function of the first order $\theta(z)$,

$$F(z) = \frac{a + b\theta(z) + c[\theta(z)]^2 + \ldots + g[\theta(z)]^n}{a' + b'\theta(z) + c'[\theta(z)]^2 + \ldots + g'[\theta(z)]^n},$$

in which $a, a', g, g'$ are all $\neq 0$, and therefore $f(z)$ *is again a rational function of* $\theta(z)$, namely,

$$f(z) = \frac{a_1 + b_1\theta(z) + \ldots + g_1[\theta(z)]^n}{a_2 + b_2\theta(z) + \ldots + g_2[\theta(z)]^n},$$

where at least one of the coefficients $a_1$ and $a_2$ and at least one of the coefficients $g_1$ and $g_2$ are $\neq 0$, because otherwise the order of the function $f(z)$ would be less than $n$.

We have seen (§16) that every circular function of the first order can be expressed in form of a linear fractional function of any other circular function of the first order. Hence this

THEOREM.—*Every circular function of the $n^{th}$ order $f(z)$ can be represented as a rational function of any assigned circular function of the first order $\phi(z)$*

$$f(z) = \frac{a + b\phi(z) + \ldots + g[\phi(z)]^n}{a' + b'\phi(z) + \ldots + g'[\phi(z)]^n}, \tag{20}$$

*where at least one of the coefficients $a$ and $a'$ and one of the coefficients $g$ and $g'$ are* $\neq 0$.

The coefficients $g$ and $g'$ cannot both vanish, because then $f(z)$ would be a circular function of the order $n-1$ at the most. For a similar reason $a$ and $a'$ cannot both vanish, because otherwise we could cancel the common factor $\phi(z)$ in the numerator and in the denominator of the expression (20), and then the order of $f(z)$ would be again $n-1$ at the most.

COROLLARY.—*Circular functions admit an algebraic addition theorem.*

In fact let us put $\phi(z) = u_\omega$ in (20), then

$$f(z) = \frac{P(u_\omega)}{Q(u_\omega)}, \tag{20'}$$

where $P(u_\omega)$ and $Q(u_\omega)$ are polynomials in $u_\omega$. If we denote $\phi(t)$ by $v_\omega$ we shall have in a similar way

$$f(t) = \frac{P(v_\omega)}{Q(v_\omega)}, \tag{20''}$$

$$f(z+t) = \frac{P(u_\omega v_\omega)}{Q(u_\omega v_\omega)}. \tag{20'''}$$

Eliminating $u_\omega$ and $v_\omega$ between the three equations (20)′, (20)″, (20)‴, we obtain an expression of the form

$$F[f(z),\ f(t),\ f(z+t)]=0,$$

where $F$ denotes an algebraic function. Q. E. D.

§21. Let us take again $\phi(z)\equiv u_\omega$ in formula (20). According as $a\neq 0$ or $a'\neq 0$ we have the two forms:

$$f(z)=\frac{a_0+a_1u_\omega+a_2u_\omega^2+\ldots\ldots+a_pu_\omega^p}{b_\kappa u_\omega^\kappa+b_{\kappa+1}u_\omega^{\kappa+1}+\ldots\ldots+b_qu_\omega^q}, \tag{21}$$

$$f(z)=\frac{a_\kappa u_\omega^\kappa+a_{\kappa+1}u_\omega^{\kappa+1}+\ldots\ldots+a_pu_\omega^p}{b_0+b_1u_\omega+b_2u_\omega^2+\ldots\ldots+b_qu_\omega^q}, \tag{22}$$

where at least one of the numbers $p$ or $q$ is equal to $n$, i. e. to the order of the circular function $f(z)$.

A first result of these expressions is that *if a characteristic limit of a circular function is infinite (or zero), then the function has an exponential infinity (or an exponential zero) of some order at this limit.* In fact, according as $p\geq q$ or $p\leq q$, formulas (21) and (22) show that

$$(p\geq q)\quad \lim_{\substack{z=\infty\\ u_\omega=\infty}}\left[\frac{f(z)}{u_\omega^{p-q}}\right]=\frac{a_p}{b_q}\begin{cases}\neq 0\\ \neq\infty\end{cases},$$

$$(q\geq p)\quad \lim_{\substack{z=\infty\\ u_\omega=\infty}}[u_\omega^{q-p}f(z)]=\frac{a_p}{b_q}\begin{cases}\neq 0\\ \neq\infty\end{cases},$$

and according as we have the form (21) or the form (22),

$$[\text{form } (21)]\quad \lim_{\substack{z=\infty\\ u_\omega=0}}[u_\omega^\kappa f(z)]=\frac{a_0}{b_\kappa}\begin{cases}\neq 0\\ \neq\infty\end{cases},$$

$$[\text{form } (22)]\quad \lim_{\substack{z=\infty\\ u_\omega=0}}\left[\frac{f(z)}{u_\omega^\kappa}\right]=\frac{a_\kappa}{b_0}\begin{cases}\neq 0\\ \neq\infty\end{cases}.$$

The order of the exponential infinity or of the exponential zero in the negative direction is always $|p-q|$, the order of the exponential infinity or of the exponential zero in the positive direction always $\kappa$. If $p>q$, then the characteristic limit in the negative direction is an exponential infinity (of the order $p-q$), and if $p<q$ then it is an exponential zero (of the order $q-p$). When the function $f(z)$ has the form (21) then the characteristic limit in the positive

direction is an exponential infinity (of the order $\varkappa$), and if $f(z)$ has the form (22) then this characteristic limit is an exponential zero (of the order $\varkappa$).

If one of the characteristic limits of $f(z)$ is to be finite and $\neq 0$, we must have either $p = q = n$ (if the finite characteristic limit is in the negative direction) or $\varkappa = 0$ (if it is to be in the positive direction), and if both characteristic limits are to be finite we must have

$$p = q = n\,; \quad \varkappa = 0\,,$$

and we thus obtain again the theorem expressed by the equation (19)′ (§20).

The vanishing points and poles of $f(z)$ in a primitive region are given respectively by the equations

$$\text{[form (21)]} \quad \begin{cases} a_0 + a_1 u_\omega \quad + \ldots\ldots + a_p u_\omega^p \quad = 0\,, \\ b_\varkappa + b_{\varkappa+1} u_\omega + \ldots\ldots + b_q u_\omega^{q-\varkappa} = 0\,; \end{cases}$$

$$\text{[form (22)]} \quad \begin{cases} a_\varkappa + a_{\varkappa+1} u_\omega + \ldots\ldots + a_p u_\omega^{p-\varkappa} = 0\,, \\ b_0 + b_1 u_\omega \quad + \ldots\ldots + b_q u_\omega^q \quad = 0\,. \end{cases}$$

To each root of these equations corresponds one determinate point in a primitive region. Hence we have in the first case $p$ vanishing points and $q - \varkappa$ poles; in the second, $p - \varkappa$ vanishing points and $q$ poles. Combining these results with those obtained with regard to the exponential infinities and zeros, we shall have, if $p \geqq q$ (then $p = n$),

[form (21)] $\begin{cases} p \text{ vanishing points.} \\ q - \varkappa \text{ poles; } (p - q) + \varkappa \text{ exponential infinities.} \end{cases}$

[form (22)] $\begin{cases} p - \varkappa \text{ vanishing points; } \varkappa \text{ exponential zeros.} \\ q \text{ poles; } p - q \text{ exponential infinities,} \end{cases}$

and if $p \leqq q$ (then $q = n$),

[form (21)] $\begin{cases} p \text{ vanishing points; } q - p \text{ exponential zeros.} \\ q - \varkappa \text{ poles; } \varkappa \text{ exponential infinities.} \end{cases}$

[form (22)] $\begin{cases} p - \varkappa \text{ vanishing points; } \varkappa + (q - p) \text{ exponential zeros.} \\ q \text{ poles.} \end{cases}$

This table shows that in all cases *the circular function $f(z)$ of the $n^{th}$ order vanishes as many times as it becomes infinite in a primitive region, namely, n times.*

More generally, we can now enunciate the following

THEOREM.—*A circular function of the $n^{th}$ order assumes any arbitrarily assigned value exactly n times in a primitive region.*

Indeed to prove this theorem we only need to consider the function $\phi(z) = f(z) - N$ where $N$ is any arbitrarily assigned number, and to apply to the function $\phi(z)$ the results previously obtained.

REMARK.—If $f(z)$ is finite and $\neq 0$ in the finite portion of a primitive region, then it has the form

$$f(z) = Cu_{\omega}^{\pm n},$$

as can be readily seen from the formulas (21) and (22).

§22. Let us put in evidence the vanishing points and poles of $f(z)$ in a primitive region. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be the vanishing points and $\lambda_1, \lambda_2, \ldots, \lambda_p$ their respective orders; $\beta_1, \beta_2, \ldots, \beta_q$ the poles and $\mu_1, \mu_2, \ldots, \mu_q$ their respective orders, in a primitive region of $f(z)$. We will denote the value of $u_{\omega}$ at a vanishing point $\alpha_{\kappa}$ by $v_{\kappa}$ and at a pole $\beta_{\kappa}$ by $w_{\kappa}$. Then both forms (21) and (22) of the function $f(z)$ will be contained in the formula

$$f(z) = cu_{\omega}^{\nu} \frac{(u_{\omega} - v_1)^{\lambda_1} (u_{\omega} - v_2)^{\lambda_2} \ldots (u_{\omega} - v_p)^{\lambda_p}}{(u_{\omega} - w_1)^{\mu_1} (u_{\omega} - w_2)^{\mu_2} \ldots (u_{\omega} - w_q)^{\mu_q}}, \tag{23}$$

where $\nu$ may be a positive or negative integer or zero.*

Let us put

$$\rho = \nu + \sum_1^p \lambda_{\kappa} - \sum_1^q \mu_{\kappa}. \tag{24}$$

Then, according as $\rho > 0$ or $\rho < 0$, the characteristic limit of $f(z)$ in the negative direction ($u_{\omega} = \infty$) will be an exponential infinity or an exponential zero, the order of multiplicity being always equal to $|\rho|$. And according as $\nu > 0$ or $\nu < 0$, the characteristic limit of $f(z)$ in the positive direction ($u_{\omega} = 0$) will be an exponential zero or an exponential infinity, the order of multiplicity being always equal to $|\nu|$.

If we differentiate the log of (23) with respect to $z$, we obtain

$$\frac{f'(z)}{f(z)} = \frac{2\pi i}{\omega}\left\{\nu + \sum_1^p \frac{\lambda_{\kappa} u_{\omega}}{u_{\omega} - v_{\kappa}} - \sum_1^q \frac{\mu_{\kappa} u_{\omega}}{u_{\omega} - w_{\kappa}}\right\}, \tag{25}$$

* It will be noticed that the function (23) is a circular function when $\nu$ is any rational number, but $\nu$ must be an integer or zero if the period of $f(z)$ is to be $\omega$. If we had $\nu = \frac{\kappa}{s}$, where $\kappa$ and $s$ are integers and the fraction is irreducible, then the period of $f(z)$ would be $s\omega$.

and making $z$ tend to infinity in the negative, respectively positive direction

$$\lim_{\substack{z=\infty \\ u_\omega=\infty}} \left[\frac{f'(z)}{f(z)}\right] = \frac{2\pi i}{\omega}\rho\,; \quad \lim_{\substack{z=\infty \\ u_\omega=0}} \left[\frac{f'(z)}{f(z)}\right] = \frac{2\pi i}{\omega}\nu\,.$$

A first result from these formulas is that the function $f'(z)$, which is evidently simply periodic like $f(z)$, tends to a determinate limit when $z$ tends to infinity in a fixed direction while remaining in a primitive region. Hence (§10) this

THEOREM I.—*The derivative $f'(z)$ of a circular function $f(z)$ admitting the period $\omega$ is also a circular function having the same period.*

We have seen that when a characteristic limit of a circular function is finite and $\neq 0$, then $\rho = 0$ (if it is the characteristic limit in the negative direction) or $\nu = 0$ (if it is the characteristic limit in the positive direction), and that $\rho = \nu = 0$ if both characteristic limits of $f(z)$ are finite and $\neq 0$. Hence, by (26), we have the

THEOREM II.—*When a characteristic limit of a circular function $f(z)$ is finite and $\neq 0$, then the corresponding* characteristic limit of the derivative function $f'(z)$ vanishes, i. e. it is an exponential zero.*

This exponential zero is, in general, of the first order, but it may be of an order higher than the first. In fact formula (25) gives according as $f_2$ (characteristic limit of $f(z)$ in the negative direction) or $f_1$ (characteristic limit in the positive direction) is finite and $\neq 0$,

$$f_2\left\{\begin{matrix}\neq 0 \\ \neq \infty\end{matrix}\right. \quad \frac{f'(z)}{f(z)} = \frac{2\pi i}{\omega}\left\{\sum_1^p \frac{\lambda_\kappa v_\kappa}{u_\omega - v_\kappa} - \sum_1^q \frac{\mu_\kappa w_\kappa}{u_\omega - w_\kappa}\right\},$$
$(\rho = 0)$

$$f_1\left\{\begin{matrix}\neq 0 \\ \neq \infty\end{matrix}\right. \quad \frac{f'(z)}{f(z)} = \frac{2\pi i}{\omega}\left\{\sum_1^p \frac{\lambda_\kappa u_\omega}{u_\omega - v_\kappa} - \sum_1^q \frac{\mu_\kappa u_\omega}{u_\omega - w_\kappa}\right\}.$$
$(\nu = 0)$

In the first case we have

$$\lim_{\substack{z=\infty \\ u_\omega=\infty}} [f'(z)\,u_\omega] = \frac{2\pi i}{\omega} f_2 \left\{\sum_1^p \lambda_\kappa v_\kappa - \sum_1^q \mu_\kappa w_\kappa\right\}.$$

* i. e. in the same direction.

In the second,

$$\lim_{\substack{z=\infty \\ u_\omega=0}} \left[\frac{f'(z)}{u_\omega}\right] = \frac{2\pi i}{\omega} f_1 \left\{-\sum_1^p \frac{\lambda_\kappa}{v_\kappa} + \sum_1^q \frac{\mu_\kappa}{w_\kappa}\right\}.$$

Hence, provided that $\sum_1^p \lambda_\kappa v_\kappa \neq \sum_1^q \mu_\kappa w_\kappa$ or that $\sum_1^p \frac{\lambda_\kappa}{v_\kappa} \neq \sum_1^q \frac{\mu_\kappa}{w_\kappa}$, the corresponding characteristic limit of $f'(z)$ will be an exponential zero of the first order. But if in the first case $\sum \lambda_\kappa v_\kappa = \sum \mu_\kappa w_\kappa$ or if in the second $\sum \frac{\lambda_\kappa}{v_\kappa} = \sum \frac{\mu_\kappa}{w_\kappa}$, then the corresponding characteristic limit of $f'(z)$ will be an exponential zero of an order higher than the first. That *it will in all cases be an exponential zero of a certain order* follows from the fact that $f'(z)$ is a circular function (§21).

Suppose now that one of the characteristic limits of $f(z)$ is not finite and $\neq 0$. Then either $\rho \neq 0$ or $\nu \neq 0$. Formulas (26) show that in this case the corresponding characteristic limit of $f'(z)$ is an exponential infinity or an exponential zero, according as the characteristic limit of $f(z)$ is infinite or zero; moreover, the order of multiplicity of the infinity or zero is the same for the function and for its derivative. Hence this

THEOREM III.—*If a characteristic limit of a circular function is an exponential infinity* (or an exponential zero) *of an order* $m$, *then the corresponding characteristic limit of the derivative function is also an exponential infinity* (or an exponential zero) *of the same order* $m$.

§23. The last theorem enables us to determine the order of the derivative function $f'(z)$ given the order $n$ of the function $f(z)$ and the number $q$ of its *distinct* poles in a primitive region. In fact, let $\beta_1, \beta_2, \ldots, \beta_q$ be the poles of $f(z)$ in a primitive region and $\mu_1, \mu_2, \ldots, \mu_q$ their respective orders of multiplicity. Let also $\mu$ and $\mu'$ be the respective orders of the exponential infinities of $f(z)$ in the positive and in the negative direction.

Each pole $\beta_i$ of $f(z)$ is a pole of the order $\mu_i + 1$ of the derivative function $f'(z)$. And the characteristic limits of $f'(z)$ are exponential infinities of the orders $\mu$ and $\mu'$ respectively. The function $f'(z)$ has no other infinities in the same primitive region. Hence it becomes infinite exactly $\sum_1^q \mu_\kappa + \mu + \mu' + q$ times in a primitive region. But $\sum_1 \mu_\kappa + \mu + \mu' = n$, hence the order of $f'(z)$ is $n+q$. We have thus the

THEOREM.—*If $q$ denote the number of distinct poles of a circular function $f(z)$ of the $n^{th}$ order in a primitive region, then the order of the derivative function $f'(z)$ is $n+q$.*

§24. MÉRAY'S THEOREM.*—*If the two characteristic limits $f_1$ and $f_2$ of a circular function $f(z)$ are finite, then the sum of its residues with respect to points in a primitive region is equal to*

$$\frac{\omega}{2\pi i}(f_2 - f_1),$$

*where as before $f_1$ is the characteristic limit in the positive direction ($u_\infty = 0$).*

Let us take two points $a$ and $b$ on a line $L_{\kappa-1}$ and their congruents on the line $L_\kappa$ as on the figure of §15. It is readily seen that the positive direction of the line $L_{\kappa-1}$ is that from $b$ to $a$. If then we take the points $a$ and $b$ sufficiently far from the origin of the plane we shall have

$$|f(z) - f_1| < \varepsilon \tag{27}$$

for all the points on the line $aa'$, and

$$|f(z) - f_2| < \varepsilon \tag{28}$$

for those of the line $bb'$.

Now, we know that the sum of the residues of $f(z)$ with respect to all points within the parallelogram $(abb'a')$ is equal to the integral

$$J = \frac{1}{2\pi i}\int f(z)\,dz$$

taken along the boundary of this parallelogram in the positive direction, which is obviously *opposed* to the positive direction of the line $L$. We then have

$$2\pi i J = \int_a^b f(z)\,dz + \int_b^{b'} f(z)\,dz + \int_{b'}^{a'} f(z)\,dz + \int_{a'}^{a} f(z)\,dz.$$

*Op. cit. vol. I, p. 274. I have given this theorem in my lectures on the Theory of Functions at the Johns Hopkins University in 1895 before the appearance of the second volume of M. Méray's treatise.

The first and the third of the integrals on the right-hand side of this equation evidently cancel each other. At the same time we have, by (27) and (28),

$$\left| \int_b^{b'} f(z)\, dz - \omega f_2 \right| < \varepsilon\omega ,$$
$$\left| \int_a^{a'} f(z)\, dz - \omega f_1 \right| < \varepsilon\omega ,$$

and therefore if we let the points $a$ and $b$ tend to infinity in the positive, respectively negative direction, we obtain in the limit

$$2\pi i J = \omega\,(f_2 - f_1). \qquad \text{Q. E. D.}$$

§25. THEOREM.—*Given a circular function $f(z)$ of the $n^{th}$ order and with the period $\omega$, let $A$ and $B$ be any two numbers other than the characteristic limits of $f(z)$; if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be respectively the roots of the equations $f(z) = A$ and $f(z) = B$ in a primitive region, we shall have*

$$\sum_1^n a_\kappa - \sum_1^n b_\kappa \equiv \frac{\omega}{2\pi i} \log \frac{(f_1 - B)(f_2 - A)}{(f_1 - A)(f_2 - B)} \ (\text{mod. } \omega). \tag{29}$$

Let us first assume that $f_1$ and $f_2$ are finite, and let $C$ be any number other than $f_1$ and $f_2$. Consider the integral

$$\frac{1}{2\pi i} \int \frac{z f'(z)}{f(z) - C}\, dz$$

taken along the boundary of the same parallelogram as in the preceding paragraph. The path of integration being broken into four parts as before, we see first of all that

$$\int_a^b \frac{z f'(z)}{f(z) - C}\, dz + \int_{b'}^{a'} \frac{z f'(z)}{f(z) - C}\, dz = \int_a^b \frac{z f'(z)}{f(z) - C}\, dz - \int_a^b \frac{(z+\omega) f'(z)}{f(z) - C}\, dz$$
$$= -\omega \int_a^b \frac{f'(z)}{f(z) - C}\, dz = -\omega \lg \left[ \frac{f(b) - C}{f(a) - C} \right] \pm 2\kappa\pi\omega i .$$

As to the remaining two integrals, they vanish separately, when $a$ and $b$ tend to infinity in the positive, respectively negative direction. In fact, the characteristic limits of $f'(z)$ being exponential zeros (§22), it is clear that

$$\lim_{\substack{z=\infty \\ u_\omega = 0}} [z f'(z)] = 0 ; \quad \lim_{\substack{z=\infty \\ u_\omega = \infty}} [z f'(z)] = 0 ,$$

therefore, provided the points $a$ and $b$ are sufficiently far from the origin, we shall have

$$\left|\frac{zf'(z)}{f(z)-C}\right| < \varepsilon$$

for all points on the lines $aa'$ and $bb'$, and the two integrals

$$\int_a^{a'} \frac{zf'(z)}{f(z)-C}\,dz\,; \quad \int_b^{b'} \frac{zf'(z)}{f(z)-C}\,dz$$

vanish in the limit. Hence in the limit

$$\frac{1}{2\pi i}\int \frac{zf'(z)}{f(z)-C}\,dz = \frac{\omega}{2\pi i}\log\left(\frac{f_1-C}{f_2-C}\right) \pm \varkappa\omega.$$

On the other hand, we know that the integral on the left-hand side of the last equation is equal to

$$-\sum_1^n c_\kappa + \sum_1^n \beta_\kappa,$$

where $c_1, c_2, \ldots, c_n$ are the roots of the equation $f(z) = C$ in a primitive region and $\beta_1, \beta_2, \ldots, \beta_n$ the poles of $f(z)$ in the same region. Hence

$$-\sum_1^n c_\kappa + \sum_1^n \beta_\kappa \equiv \frac{\omega}{2\pi i}\log\left(\frac{f_1-C}{f_2-C}\right) \pmod{\omega}.$$

Suppose now that one or both of the characteristic limits of $f(z)$ may be infinite. Let then $A$ and $B$ be numbers other than $f_1$ and $f_2$, and consider the circular function of the $n^{\text{th}}$ order

$$\phi(z) = \frac{1}{f(z)-A},$$

whose characteristic limits are finite. The poles of $\phi(z)$ in a primitive region are the roots of the equation $f(z) = A$ in the same region, i. e. the points $a_1, a_2, \ldots, a_n$; and the roots of the equation $\phi(z) = \dfrac{1}{B-A}$ are the roots of the equation $f(z) = B$, i. e. the points $b_1, b_2, \ldots, b_n$. Then by the proposition just proved

$$\sum_1^n {}_\kappa a_\kappa - \sum_1^n {}_\kappa b_\kappa \equiv \frac{\omega}{2\pi i} \log \left\{ \frac{\dfrac{1}{f_1 - A} - \dfrac{1}{B - A}}{\dfrac{1}{f_2 - A} - \dfrac{1}{B - A}} \right\} \pmod{\omega}$$

$$\equiv \frac{\omega}{2\pi i} \log \frac{(f_1 - B)(f_2 - A)}{(f_1 - A)(f_2 - B)} \pmod{\omega}$$

Q. E. D.)

COROLLARY I.—*If the characteristic limits of a circular function are equal to each other, then*

$$\sum_1^n {}_\kappa a_\kappa \equiv \sum_1^n {}_\kappa b_\kappa \pmod{\omega}.$$

The case when both characteristic limits are infinite is here included.

COROLLARY II.—*If only one of the characteristic limits of $f(z)$ is infinite and the other be denoted by $f'$, then*

$$\sum_1^n {}_\kappa a_\kappa - \sum_1^n {}_\kappa b_\kappa \equiv \pm \frac{\omega}{2\pi i} \log \left( \frac{f' - A}{f' - B} \right) \pmod{\omega} \qquad (30)$$

*the upper or lower sign to be taken according as* $\lim_{z=\infty} f(z) = \infty$ *in the positive or in the negative direction.*

§26. Let us now put $f(z) = \zeta$, where $\zeta$ is a variable quantity, and let $z_1, z_2, \ldots, z_n$ be the roots of this equation. These roots are functions of $\zeta$. By formula (29)

$$\sum_1^n {}_\kappa z_\kappa = \frac{\omega}{2\pi i} \log \left( \frac{f_2 - \zeta}{f_1 - \zeta} \right) + \text{constant},$$

provided $\zeta \neq f_1$ or $f_2$. If we differentiate this formula with respect to $\zeta$, we obtain

$$\sum_1^n {}_\kappa \frac{dz_\kappa}{d\zeta} = \frac{\omega}{2\pi i} \left\{ \frac{1}{f_1 - \zeta} - \frac{1}{f_2 - \zeta} \right\}.$$

But $\dfrac{dz_\kappa}{d\zeta}$ being one of the values of $\dfrac{dz}{d\zeta}$ corresponding to the value of $\zeta$, we have

$$\sum_1^n {}_\kappa \frac{dz_\kappa}{d\zeta} = \sum_1^n {}_\kappa \frac{1}{f'(z_\kappa)}.$$

Hence this

THEOREM.—*Given a circular function of the $n^{th}$ order $f(z)$ with the period $\omega$, let $z_1, z_2, \ldots, z_n$ be the points in a primitive region at which the function assumes the same value $\zeta \neq f_1$ or $f_2$, then*

$$\sum_{1}^{n}{}_{\kappa} \frac{1}{f'(z_\kappa)} = \frac{\omega}{2\pi i}\left\{\frac{1}{f_1-\zeta} - \frac{1}{f_2-\zeta}\right\}. \tag{31}$$

COROLLARY.—*If the characteristic limits of the function $f(z)$ are equal to each other or infinite, then*

$$\sum_{1}^{n}{}_{\kappa} \frac{1}{f'(z_\kappa)} = 0,$$

*$z_1, z_2, \ldots, z_n$ being the points in a primitive region at which the function $f(z)$ assumes the same value other than the value of the characteristic limits.*

§27. THEOREM.—*If two circular functions $f(z)$ and $\phi(z)$ have the same period, then they are connected by an algebraic equation.*

In fact, these functions can be presented in the form of rational functions of $u_\omega$, i. e. (§20),

$$f(z) = \frac{P(u_\omega)}{Q(u_\omega)}, \tag{32}$$

$$\phi(z) = \frac{P_1(u_\omega)}{Q_1(u_\omega)}. \tag{33}$$

The elimination of $u_\omega$ between the equations (32), (33) gives an irreducible algebraic equation of the form

$$F(f(z),\ \phi(z)) = 0, \tag{34}$$

which proves the theorem. The degree of this equation is, in general, $n$ in $f(z)$ and $m$ in $\phi(z)$, if $m$ and $n$ are the orders of the circular functions $f(z)$ and $\phi(z)$ respectively.* This follows at once from the process of elimination, but there is

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* Let

$$f(z) = \frac{a_0 + a_1 u_\omega + \ldots + a_p u_\omega^p}{b_0 + b_1 u_\omega + \ldots + b_q u_\omega^q}; \quad \varphi(z) = \frac{a_0' + a_1' u_\omega + \ldots + a_{p'}' u_\omega^{p'}}{b_0' + b_1' u_\omega + \ldots + b_{q'}' u_\omega^{q'}};$$

where at least one of the numbers $p$ or $q$ is $= m$; and at least one of the numbers $p'$ or $q'$ is $= n$. If, then, we put for the sake of brevity

$$b_\kappa f(z) - a_\kappa = x_\kappa; \quad b_\kappa' \phi(z)\, b - a_\kappa' = y_\kappa,$$

another way of arriving to this result. To each value of $\phi(z)$ correspond within a primitive region $n$ points, and in the whole plane an infinity of points, which may be all classified into $n$ groups, each point of a group being congruent with any other point of the same group. To each of these groups corresponds a single value of $f(z)$. Hence to each value of $\phi(z)$ correspond $n$ values of $f(z)$. Likewise to each value of $f(z)$ correspond $m$ values of $\phi(z)$. In general these $n$ values of $f(z)$, respectively the $m$ values of $\phi(z)$ will be distinct, and therefore the irreducible equation will in general be of the degree $n$ in $f(z)$ and of the degree $m$ in $\phi(z)$. But the degrees of the irreducible equation in $f(z)$ and $\phi(z)$ will be lesser if to several groups of points $z$ always corresponds only one value of $f(z)$ or $\phi(z)$. It can be readily shown that in such cases the degrees of the irreducible equation in $f(z)$ and $\phi(z)$ are $n_1 = \frac{n}{D}$ and $m_1 = \frac{m}{D}$ respectively, where $D$ is a common divisor of the numbers $n$ and $m$,* and is equal to the

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the result of the elimination of $u_\omega$ between the two given equations may be written as follows:

$$\begin{vmatrix} x_0 & x_1 & x_2 & \cdots & x_m & 0 & 0 & \cdots & 0 \\ 0 & x_0 & x_1 & \cdots & \cdots & x_m & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & x_0 & \cdots & \cdots & \cdots & \cdots & x_m \\ y_0 & y_1 & \cdots & y_{n-1} & y_n & 0 & 0 & \cdots & 0 \\ 0 & y_0 & \cdots & \cdots & \cdots & y_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & y_0 & \cdots & \cdots & \cdots & y_n \end{vmatrix} = 0. \qquad (34)'$$

The letters $x_0, \ldots, x$ enter only in the first $n$ lines of this determinant of the $(m+n)^{\text{th}}$ order; the letters $y_0, \ldots, y_n$ only in the last $m$ lines. Therefore this determinant is of the degree $n$ in the function $f(z)$ and of the degree $m$ in the function $\phi(z)$. The equation (34)′ being only another form of equation (34), it is clear that the latter will be of the degrees $n$ and $m$ in $f(z)$ and $\phi(z)$ respectively.

To operate the elimination it will, in general, be necessary to add the following $m+n-2$ equations

$$u_\omega^\kappa [Q(u_\omega) f(z) - P(u_\omega)] = 0,$$
$$\kappa \equiv 1, 2, \ldots, n-1$$
$$u_\omega^\kappa [Q_1(u_\omega) \phi(z) - P_1(u_\omega)] = 0,$$
$$\kappa \equiv 1, 2, \ldots, m-1.$$

to the equations (32) and (33), in which case we obtain equation (34)′ as the result of the elimination. But in particular cases the number of subsidiary equations necessary to operate the elimination may be less than $m+n-2$. Then the degree of the equation (34) in $f(z)$ and $\phi(z)$ may be less than $n$ and $m$ respectively.

*See Méray, op. cit. vol. II, p. 359.

number of groups of points $z$ which always correspond to the same value of $f(z)$ or $\phi(z)$.

COROLLARY.—*Every circular function $f(z)$ of the $n^{th}$ order is connected with its derivative function $f'(z)$ by an algebraic equation* (§22, Theorem I). *The degree of this equation is $n$ in $f'(z)$ and $n+q$ in $f(z)$, $q$ being the number of distinct poles of $f(z)$ in a primitive region* (§23).

§28. Let us for the sake of brevity write $w$ for $f(z)$; $w'$ for $f'(z)$; and $W$ for any polynomial in $w$. Then we may write the irreducible algebraic equation connecting the circular function $f(z)$ with its derivative as follows:

$$W_0 w'^n + W_1 w'^{n-1} + \dots + W_{n-1} w' + W_n = 0 \tag{35}$$

where at least one of the polynomials $W_0, W_1, \dots, W_n$ is of the degree $n+q$. First of all it is clear that $W_0 =$ constant, because otherwise the equation $W_0 = 0$ would determine points such that at these points $w' = \infty$ while $w$ is finite, which is impossible. We may therefore assume that $W_0 = 1$. Further, provided $w \neq f_1$ or $f_2$, we have

$$\frac{W_{n-1}}{W_n} = \frac{\omega}{2\pi i}\left\{\frac{1}{f_2 - w} - \frac{1}{f_1 - w}\right\}. \tag{36}$$

This follows from the relation

$$\frac{W_{n-1}}{W_n} = -\sum_1^n \frac{1}{w'_\kappa} = -\sum_1^n \frac{1}{f'(z_\kappa)}$$

and from formula (31).

§29. Without going into a detailed study of the problem of inversion, the solution of the following particular cases will be given here.

PROBLEM I.—*Find all circular functions of the first order.*

Equation (35) becomes in this case either

$$w' = c(w - \alpha)(w - \beta) \tag{37}$$

when $q = 1$ (*the characteristic limits of $w$ are finite*); or

$$w' = aw + b \tag{38}$$

when $q = 0$ (*one of the characteristic limits of $w$ is infinite*).

All circular functions of the first order are therefore the inverse of the functions defined by the integrals

$$z=\int\frac{dw}{c\,(w-\alpha)(w-\beta)}+c'; \quad z=\int\frac{dw}{aw+b}+c. \tag{39}$$

The *moduli of periodicity* $\omega$ of these integrals (which are the periods of the inverse functions) are, as easily verified,

$$\omega=\frac{2\pi i}{c\,(\alpha-\beta)}; \quad \omega=\frac{2\pi i}{a}$$

respectively. The functions $w$ are then

$$w\equiv f(z)=\frac{\alpha-A\beta e^{\frac{2\pi i}{\omega}z}}{1-Ae^{\frac{2\pi i}{\omega}z}}, \tag{40}$$

$$w\equiv\phi\,(z)=Ae^{\frac{2\pi i}{\omega}z}+B \tag{41}$$

respectively. The characteristic limits of the circular function $f(z)$ are

$$f_1=\alpha; \ f_2=\beta,$$

and those of the function $\phi\,(z)$

$$f_1=B=-\frac{b}{a}; \ f_2=\infty.$$

If we put $\alpha=i$; $\beta=-i$; $c=1$ in (37), then $f(z)$ is the inverse of the integral

$$\int\frac{dw}{w^2+1},$$

which may be taken as the definition of the function tg $z$. In this case formula (40) takes the form

$$w\equiv f(z)=A\ \mathrm{tg}\left(\frac{\pi}{\omega}z+B\right)+C, \tag{42}$$

and therefore we may say that *the functions* $e^z$ *and* tg $z$ *are the two types of the circular functions of the first order.* All such functions are comprised in one or the other of the forms (41) and (42).

PROBLEM II.—*Find all circular functions of the second order whose characteristic limits are infinite.*

In this case $n=2$; $q=0$, and taking into consideration that by formula (36) $W_1=0$, equation (35) reduces to

$$w'^2 + W_2 = 0.$$

Hence all circular functions of the second order with infinite characteristic limits are the inverse of the functions defined by the integral

$$\int \frac{dw}{\sqrt{a+bw+cw^2}}. \tag{43}$$

If we put here $b=0$; $a=1$; $c=-1$, we may take this for the definition of the function $\sin z$. Then all the inverse of the function defined by the integral (43) will be comprised in the general formula

$$w \equiv f(z) = A \sin\left(\frac{2\pi}{\omega} z + B\right) + C. \tag{44}$$

REMARK.—It will be noticed that the characteristic limits of a circular function are the essentially singular points of the inverse function.

§30. To conclude, we will give some general propositions with regard to circular functions of the second order to show their analogy with elliptic functions of the second order.

THEOREM I.—*Let $f(z)$ be a circular function of the second order whose characteristic limits are equal to each other and to the number $N$. If $\gamma_1, \gamma_2$ be the points in a primitive region at which $f(z)$ assumes the same value $\Gamma \neq N$, then*

$$f(z) = f(\gamma_1 + \gamma_2 - z). \tag{45}$$

For we have in this case $z_1 + z_2 \equiv \gamma_1 + \gamma_2 \pmod{\omega}$, where $z_1, z_2$ are the points in a primitiveregion at which the function assumes the same value $f(z)$ (§25, Corollary I), and therefore $f(z_1) = f(z_2) = f(\gamma_1 + \gamma_2 - z_1)$, where $z_1$ may be any point.

COROLLARY.—*If we take for the origin of the plane a point $\frac{1}{2}(\gamma_1 + \gamma_2)$, then the function of Theorem 1 will be an even function, i. e. $f(-z) = f(z)$.*

THEOREM II.—*If $N \neq \infty$, then the function of Theorem I has two distinct poles $\beta_1, \beta_2$ of the first order or one pole $\beta$ of the second order in a primitive region. In the first case $f'(z)$ is a circular function of the fourth order having two exponential* *zeros of the first order and two vanishing points of the first order in a primitive region, namely, the points* $\frac{1}{2}(\beta_1+\beta_2)$, $\frac{1}{2}(\beta_1+\beta_2+\omega)$; *in the second case* $f'(z)$ *is of the third order and has two exponential zeros of the first order and one vanishing point of the first order in a primitive region, namely, the point* $\beta+\frac{1}{2}\omega$.

THEOREM III.—*If* $N=\infty$, *then the function of Theorem I has no poles in a primitive region, and* $f'(z)$ *is also a circular function of the second order.*

*Let then* $\alpha_1$ *and* $\alpha_2$ *be the vanishing points of* $f(z)$ *in a primitive region. The vanishing points of* $f'(z)$ *in a primitive region will be the points* $\frac{1}{2}(\alpha_1+\alpha_2)$ *and* $\frac{1}{2}(\alpha_1+\alpha_2+\omega)$.

The last two propositions follow from the equation

$$f'(z)=-f'(\gamma_1+\gamma_2-z)$$

obtained by differentiation from the equation (45).

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# *Sur un problème concernant deux courbes gauches.*

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J'ai donné autrefois dans les Comptes Rendus une formule générale qui fournit la représentation sous forme explicite des surfaces réglées rapportées à leurs lignes asymptotiques. Cette formule était fondée sur la solution du problème suivant :

*Une courbe $C$ étant donnée, en trouver une autre $C_1$ qui lui corresponde point par point de sorte que le plan osculateur à chaque courbe aille passer par le point qui correspond sur l'autre au point de contact.*

Je me propose de donner ici la solution directe de ce problème.

Soit le point $M$ de la courbe $C$, dont les coordonnées $x$, $y$, $z$, $t$ sont des fonctions données d'un paramètre variable $u$.

Le point $M_1$ de $C_1$ correspondant à $M$, doit être dans le plan osculateur au point $M$ à la courbe $C$; ses coordonnées $x_1$, $y_1$, $z_1$, $t_1$ sont donc de la forme

$$\left.\begin{aligned} x_1 &= \alpha x'' + \beta x' + \gamma x, \\ y_1 &= \alpha y'' + \beta y' + \gamma y, \\ z_1 &= \alpha z'' + \beta z' + \gamma z, \\ t_1 &= \alpha t'' + \beta t' + \gamma t, \end{aligned}\right\} \tag{1}$$

où les accents désignent les dérivées prises par rapport à la variable $u$ et $\alpha$, $\beta$, $\gamma$ désignent des fonctions appropriées de $u$.

Maintenant, puisque le point $M$ doit être dans le plan osculateur en $M_1$ à la courbe $C_1$, il doit exister des relations de la forme

$$\left.\begin{aligned} \rho x_1'' + \sigma x_1' + \lambda x_1 + \mu x &= 0, \\ \rho y_1'' + \sigma y_1' + \lambda y_1 + \mu y &= 0, \\ \rho z_1'' + \sigma z_1' + \lambda z_1 + \mu z &= 0, \\ \rho t_1'' + \sigma t_1' + \lambda t_1 + \mu t &= 0. \end{aligned}\right\} \tag{2}$$

Je dis que l'on peut toujours supposer que l'on a

$$\frac{\rho}{\alpha}=\frac{\sigma}{\beta}$$

et alors, à cause de l'homogénéité des formules précédentes, nous pourrons supposer $\rho=\alpha$, $\sigma=\beta$.

Imaginons en effet, que l'on ait satisfait aux équations simultanées (1), (2) par un premier système de valeurs des fonctions $\alpha, \beta, \gamma, \lambda, \mu, \rho, \sigma$. Il est clair que l'on aura encore un système satisfaisant à ces équations en changeant $\alpha$, $\beta$, $\gamma$ en $\alpha.\varepsilon$, $\beta.\varepsilon$, $\gamma.\varepsilon$, car cela revient à multiplier $x_1$, $y_1$, $z_1$, $t_1$ par $\varepsilon$. Désignons par $\rho_1$, $\sigma_1$, $\lambda_1$, $\mu_1$ les valeurs correspondantes de $\rho$, $\sigma$, $\lambda$, $\mu$, on aura

$$\rho_1(\varepsilon x_1)''+\sigma_1(\varepsilon x_1)'+\lambda_1\varepsilon x_1+\mu_1 x=0,$$

etc., ou en développant,

$$\rho_1\varepsilon.x_1''+(2\rho_1\varepsilon'+\sigma_1\varepsilon)x_1'+(\rho_1\varepsilon''+\sigma_1\varepsilon'+\lambda_1\varepsilon)x_1+\mu_1 x=0.$$

En comparant aux relations (2) on voit qu'il faudra avoir

$$\frac{\rho_1\varepsilon}{\rho}=\frac{2\rho_1\varepsilon'+\sigma_1\varepsilon}{\sigma}=\ldots\ldots,$$

les équations que nous n'écrivons pas détermineraient $\lambda_1$, $\mu_1$. Or nous voulons avoir $\frac{\rho_1}{\alpha}=\frac{\sigma_1}{\beta}$ ; il faudra donc écrire

$$\frac{\alpha\varepsilon}{\rho}=\frac{2\alpha\varepsilon'+\beta\varepsilon}{\sigma},$$

équation différentielle qui donnera $\varepsilon$.

On voit donc que, par un choix convenable de $\varepsilon$, on pourra toujours supposer que $\rho=\alpha$, $\sigma=\beta$.

Les équations (1) conservant leur forme, les équations (2) auront alors la suivante

$$\left.\begin{aligned} \alpha x_1''+\beta x_1'+\lambda x_1+\mu x&=0,\\ \alpha y_1''+\beta y_1'+\lambda y_1+\mu y&=0,\\ \alpha z_1''+\beta z_1'+\lambda z_1+\mu z&=0,\\ \alpha t_1''+\beta t_1'+\lambda t_1+\mu t&=0,\end{aligned}\right\} \tag{2'}$$

Si, de la première des équations (1), on tire $x_1$ pour en porter la valeur dans la première des équations (2)' on trouve l'équation du quatrième ordre

$$\begin{aligned} \alpha^2.x^{(IV)} + 2\alpha(\alpha' + \beta)x''' &+ [\alpha(\alpha'' + 2\beta' + \gamma) + \beta(\alpha' + \beta) + \alpha\lambda]\,x'' \\ &+ [\alpha(\beta'' + 2\gamma') + \beta(\beta' + \gamma) + \beta\lambda]\,x' \\ &+ [\alpha\gamma'' + \beta\gamma' + \lambda\gamma + \mu]\,x = 0 \end{aligned} \tag{3}$$

et l'on voit que $y$, $z$, $t$ vérifient aussi cette équation.

Soit

$$X^{(IV)} + 4P_1X''' + 6P_2X'' + 4P_3X' + P_4X = 0, \tag{4}$$

l'équation linéaire du 4ème ordre vérifiée par $x, y, z, t$.

Cette équation doit être identique à l'équation (3). On aura donc

$$\left.\begin{aligned} &\alpha(\alpha' + \beta) &&= 2P_1.\alpha^2, \\ &\alpha(\alpha'' + 2\beta' + \gamma) + \beta(\alpha' + \beta) + \alpha\lambda &&= 6P_2.\alpha^2, \\ &\alpha(\beta'' + 2\gamma') + \beta(\beta' + \gamma) + \beta\lambda &&= 4P_3.\alpha^2, \\ &\alpha\gamma'' + \beta\gamma' + \lambda\gamma + \mu &&= P_4.\alpha^2. \end{aligned}\right\} \tag{5}$$

Ces équations se simplifient beaucoup.

En eliminant $\lambda$ entre la 2ème et la 3ème de ces équations on obtiendra le système suivant

$$\left.\begin{aligned} &\alpha' + \beta = 2P_1\alpha, \\ &\alpha\beta(\alpha'' + 2\beta' + \gamma) + \beta^2(\alpha' + \beta) - \alpha^2(\beta'' + 2\gamma') - \alpha\beta(\beta' + \gamma) \\ &\qquad = 6P_2\alpha^2\beta - 4P_3\alpha^3, \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} &\alpha(\beta'' + 2\gamma') + \beta(\beta' + \gamma) + \beta\lambda = 4P_3\alpha^2, \\ &\alpha\gamma'' + \beta\gamma' + \lambda\gamma + \mu = P_4\alpha^2. \end{aligned}\right\} \tag{7}$$

Les équations (7) donneront $\lambda$, $\mu$ quand $\alpha$, $\beta$, $\gamma$ seront connus. Ces fonctions $\alpha$, $\beta$, $\gamma$ sont liées par les équations (6), qui sont les seules dont nous ayons à nous occuper, car la connaissance de $\lambda$, $\mu$ est secondaire.

La première des équations (6) nous permettra d'eliminer $\alpha'$, $\alpha''$ de la seconde de ces équations.

La seconde de ces équations s'écrit d'abord

$$\alpha\beta(\alpha'' + \beta') + \beta^2(\alpha' + \beta) - \alpha^2(\beta'' + 2\gamma') = 6P_2\alpha^2\beta - 4P_3\alpha^3.$$

La première des équations (6) nous donne alors

$$\begin{aligned} &\alpha' + \beta = 2P_1\alpha, \\ &\alpha'' + \beta' = 2P_1\alpha' + 2P_1'\alpha = [4P_1^2 + 2P_1']\,\alpha - 2P_1\beta, \end{aligned}$$

en sorte que l'équation précédente devient

$$\alpha\beta\,[(4P_1^2+2P_1')\,\alpha-2P_1\beta]+2P_1\alpha\beta^2-\alpha^2(\beta''+2\gamma')=6P_2\alpha^2\beta-4P_3\alpha^3.$$

Après réductions on trouve

$$-\beta''+(4P_1^2+2P_1'-6P_2)\,\beta+4P_3\alpha-2\gamma'=0, \tag{8}$$

équation à joindre à la première des équations (6)

$$\alpha'+\beta=2P_1\alpha. \tag{9}$$

Posons

$$2\theta=\beta'+2\gamma, \tag{10}$$

l'équation (8) devient

$$\theta'=(2P_1^2+P_1'-3P_2)\,\beta+2P_3\alpha. \tag{11}$$

L'équation (10) nous permet de substituer la fonction $\theta$ à la fonction $\gamma$; $\alpha$, $\beta$, $\theta$ sont liées par les équations (9) et (11).

Eliminons $\beta$ entre ces équations, nous aurons

$$\theta'=2\,[P_3+P_1P_1'+2P_1^3-3P_1P_2]\,\alpha+(3P_2-P_1'-2P_1^2)\,\alpha'. \tag{12}$$

Or, en posant avec Mr. Halphen,

$$30\,V=-P_1''+3P_2'-6P_1P_1'-2P_3+6P_1P_2-4P_1^3,$$

on trouve que l'équation (12) peut s'écrire

$$\theta'=[(3P_2-P_1'-2P_1^2)\,\alpha]'-30\,V\alpha. \tag{13}$$

Soit $U$ une fonction arbitraire de $u$, et posons

$$\alpha=\frac{U'}{30\,V}, \tag{14}$$

l'équation (13) devient

$$\theta'=[(3P_2-P_1'-2P_1^2)\,\alpha]'-U',$$

ou, en intégrant et faisant rentrer dans $U$ la constante d'intégration,

$$\theta=(3P_2-P_1'-2P_1^2)\,\alpha-U. \tag{15}$$

Si l'on tient comple de la valeur de $\alpha$ on aura

$$\theta=(3P_2-P_1'-2P_1^2)\,\frac{U'}{30\,V}-U. \tag{16}$$

Les formules (14), (16) donnent les expressions générales de $\alpha$, $\theta$. L'équation (9) donnera $\beta$ et l'équation (10) donnera $\gamma$. Il viendra en définitive:

$$\left.\begin{aligned} \alpha &= \frac{U'}{30V}, \\ \beta &= 2P_1 \cdot \frac{U'}{30V} - \left(\frac{U'}{30V}\right)', \\ \gamma &= -U + (3P_2 - 2P_1' - 2P_1^2)\frac{U'}{30V} - P_1\left(\frac{U'}{30V}\right)' + \tfrac{1}{2}\left(\frac{U'}{30V}\right)''. \end{aligned}\right\} \quad (17)$$

L'expression $V$ a été introduite par Halphen au titre d'invariant projectif des courbes gauches.* Si $V = 0$, la courbe $C$ appartient par ses tangentes à un complexe lineaire. Dans ce cas, les expressions (17) tombent en défaut. Mais si l'on se reporte à l'équation (13), on voit que cette équation devient immédiatement intégrable et que $\alpha$ peut être prise comme fonction arbitraire. Les formules (17) doivent alors être remplacées par les suivantes

$$\left.\begin{aligned} \alpha &= \alpha, \\ \beta &= 2P_1\alpha - \alpha', \\ \gamma &= (3P_2 - 2P_1^2 - 2P_1')\,\alpha - P_1\alpha' + \tfrac{1}{2}\alpha''. \end{aligned}\right\} \quad (18)$$

Les formules (17) ou (18), suivant le cas, résolvent donc le problème qui nous nous étions proposé, sans aucune intégration ni quadrature.

Appliquons ceci à une cubique gauche, pour laquelle on peut supposer que $x$, $y$, $z$, $t$ sont des polynômes du 3ème degré d'un paramètre $u$. Pour une telle courbe l'équation du quatrième ordre se réduit à

$$X^{\text{IV}} = 0,$$

les $P_i$ sont tous nuls, $V$ est nul, nous sommes dans le cas des formules (18). On prendra ici

$$\alpha = \alpha, \quad \beta = -\alpha', \quad \gamma = \tfrac{1}{2}\alpha''.$$

On peut toujours supposer le tétraèdre de coordonnées tel que $x$, $y$, $z$, $t$ soient les polynômes suivants de $u$

$$x = u^3, \quad y = u^2, \quad z = u, \quad t = 1.$$

* **Acta Mathematica, t. III; Journal de l'École Polytechnique, 47ème cahier.**

Les formules (1), eu égard aux valeurs (18) de $\alpha$, $\beta$, $\gamma$ deviennent

$$\left.\begin{aligned} x_1 &= 6au - 3a'u^2 + \tfrac{1}{2}a''u^3, \\ y_1 &= 2a - 2a'u + \tfrac{1}{2}a''u^2, \\ z_1 &= \quad\; - a' + \tfrac{1}{2}a''u, \\ t_1 &= \qquad\quad \tfrac{1}{2}a''. \end{aligned}\right\} \qquad (20)$$

En prenant pour $a$ un polynôme du 3ème degré en $u$, on obtient une seconde cubique gauche $C_1$, qui correspond point par point à la première dans les conditions indiquées par l'énoncé.

### *Rapports avec les lignes asymptotiques des surfaces réglées.*

Soient $C$, $C_1$ deux lignes asymptotiques sur une surface réglée, $M$, $M_1$ les points où une même génératrice rectiligne coupe ces courbes. Le plan tangent en $M$ à la surface, qui est le plan osculateur à $C$, contient la génératrice rectiligne, il passe donc en $M_1$; de même le plan osculateur en $M_1$ à $C_1$ passe par le point $M$. Les courbes $C$, $C_1$ sont donc dans la relation indiquée plus haut, le plan osculateur en chaque point de l'une passant au point homologue sur l'autre.

Cela étant, prenons une courbe $C$ ligne asymptotique d'une surface réglée; les lignes asymptotiques $C_1$ de cette surface réglée correspondront à divers choix de la fonction $U$ dans les formules (17) (ou (18)). Nous allons chercher quel lien existe entre ces fonctions.

Soit $C_1$ une ligne asymptotique correspondant à une détermination particulière de $U$. Un point quelconque de la génératrice rectiligne de la surface sera représentée par les formules

$$\left.\begin{aligned} X &= x_1 + \zeta . x, \\ Y &= y_1 + \zeta . y, \\ Z &= z_1 + \zeta . z, \\ T &= t_1 + \zeta . t, \end{aligned}\right\} \qquad (21)$$

où $x_1$, $y_1$, $z_1$, $t_1$ sont fournis par les équations (1), tandis que les équations (2)' ont lieu; $\zeta$ est un paramètre qui fixe la position du point ($X$, $Y$, $Z$, $T$) sur la génératrice.

Observons que $X$, $Y$, $Z$, $T$, considérés comme fonctions de $u$, $\zeta$ vérifient l'équation

$$\frac{\partial^2 X}{\partial u^2} = p\frac{\partial X}{\partial u} + q\frac{\partial X}{\partial \zeta} + rX, \qquad (22)$$

où l'on a (formules (1) et (2)′)

$$p = -\frac{\beta}{\alpha},$$

$$q = \frac{-\zeta^2 + (\lambda - \gamma)\zeta - \mu}{\alpha},$$

$$r = \frac{\zeta - \lambda}{\alpha}.$$

Comme $X$, $Y$, $Z$, $T$ vérifient aussi l'équation

$$\frac{\partial^2 X}{\partial \zeta^2} = 0, \tag{23}$$

on voit, par le rapprochement des formules (22), (23), que $u$, $\zeta$ sont les paramètres des lignes asymptotiques, de la surface considerée.

On arrive donc à ce résultat que si $C$, $C_1$ sont liées par les formules (1), (2)′, la surface réglée représentée par les formules (21) admet comme asymptotiques non rectilignes les lignes $\zeta =$ constante. Les formules précédentes résolvent donc la question suivante:

*Représenter sous forme explicite les coordonnées d'un point* ($X$, $Y$, $Z$, $T$) *d'une surface réglée en fonction de deux paramètres* $\zeta$, $u$, *de sorte que* $u =$ *const. représentent les génératrices rectilignes et* $\zeta =$ *const. les lignes asymptotiques non rectilignes.*

Il est à remarquer qu'en partant d'une ligne asymptotique supposée donnée la question se résoud sans aucune intégration ni quadrature.

On sait que la recherche des asymptotiques d'une surface réglée conduit à une équation de Riccati. Si, en effet, on voulait identifier la surface réglée représentée par les formules (21) avec une surface réglée donnée à priori les calculs d'identification conduiraient à une équation de Riccati.

Il n'en résulte pas moins que les formules précédentes constituent un procédé d'étude indirecte des lignes asymptotiques des surfaces réglées, très propre à mettre en évidence des propriétés de ces lignes asymptotiques.

Supposons, par exemple, que l'on veuille toutes les surfaces réglées à asymptotiques algébriques. Il suffira de supposer que la ligne asymptotique $C$ d'où l'on part est algébrique, en sorte que $x$, $y$, $z$, $t$ et, par suite, $P_1$, $P_2$, $P_3$, $P_4$ seront des fonctions algébriques de $u$. Ensuite, il faudra prendre aussi pour $U$ une fonction algébrique de $u$.

On peut faire une remarque générale au sujet des expressions (21); si on remplace $x_1$, $y_1$, $z_1$, $t_1$ par leurs expressions (1) on trouve, par exemple,

$$X = \alpha x'' + \beta x' + (\gamma + \zeta)\, x$$

et de même pour $Y$, $Z$, $T$.

Or, reportons-nous aux formules (17) qui donnent $\alpha$, $\beta$, $\gamma$. Le changement de $\gamma$ en $\gamma + \zeta$ revient à ajouter à la fonction arbitraire $U$ une constante arbitraire $-\zeta$.

Donc les diverses lignes asymptotiques d'une même surface réglée, admettant $C$ pour asymptotique, résultent des formules (1) et (17) en prenant pour $U$ des fonctions qui ne diffèrent que par des constantes additives.

# *The Linear Vector Operator of Quaternions.*

BY JAMES BYRNIE SHAW, D. SC.

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This paper has for its object the development of the algebra of the linear vector operator, entirely from a quaternion point of view, which amounts to an extension or development of nonions. It is presumed that so much of the theory of the linear vector operator is known as is developed in works on Quaternions, as Tait, 3d edition, or Hamilton's Elements. I consider first the expression of such an operator, as $\phi$, in terms of three numbers, $a$, $b$, $c$, which depend only on the three (latent) roots of $\phi$, and a unit operator $\iota$, which depends only on the axes of $\phi$. I then consider $\phi$ as dependent on nine operators which are linearly independent, each of nullity two, three of vacuity two, and six of vacuity three. (For definitions of latent roots, axes, nullity, and vacuity, see Taber, "Theory of Matrices," Amer. Jour. Math., vol. 12, pp. 355 and 362.)

## I.

1. The function $\phi$ may be written in the form

$$\phi = a + b\iota + c\iota^2, \tag{1}$$

$a$, $b$, $c$ being scalars and $\iota$ a linear vector operator such that $\iota^3.\rho = \rho$ or $\iota^3 = 1$. The operator $\iota$ involves implicitly six scalars, since its equation $\iota^3 - 1 = 0$ determines explicitly three of the usual nine scalars involved in every general linear vector function. The proof of the existence of this form of $\phi$ will consist simply in determining explicitly each element of it. The three roots and three axes of $\phi$ are supposed known.

We have at once

$$\phi^2 = a^2 + 2bc + (c^2 + 2ab)\,\iota + (b^2 + 2ac)\,\iota^2, \tag{2}$$

$$\phi^3 = a^3 + b^3 + c^3 + 6abc + 3\,(ac^2 + ba^2 + cb^2)\,\iota + 3\,(a^2c + b^2a + c^2b)\,\iota^2. \tag{3}$$

Whence it is easy to deduce the equation

$$\phi^3 - 3a.\phi^2 + 3(a^2 - bc)\phi - (a^3 + b^3 + c^3 - 3abc) = 0. \qquad (4)$$

But the cubic in $\phi$ is

$$\phi^3 - m_2\phi^2 + m_1\phi - m = 0. \qquad (5)$$

Therefore, equating coefficients of like powers of $\phi$, and reducing,

$$a = \tfrac{1}{3} m_2 = \tfrac{1}{3}(g_1 + g_2 + g_3), \qquad (6)$$

$$bc = \tfrac{1}{9}(m_2^2 - 3m_1) = \tfrac{1}{9}(g_1^2 + g_2^2 + g_3^2 - g_1g_2 - g_2g_3 - g_3g_1), \qquad (7)$$

$$\begin{aligned} b^3 + c^3 &= \tfrac{1}{27}(27m + 2m_2^3 - 9m_1m_2) \\ &= \tfrac{1}{27}(2g_1^3 + 2g_2^3 + 2g_3^3 - 3g_1^2g_2 - 3g_1g_2^2 - 3g_2^2g_3 \\ &\qquad - 3g_2g_3^2 - 3g_3^2g_1 - 3g_3g_1^2 + 12g_1g_2g_3). \qquad (8) \end{aligned}$$

Putting $\lambda^3 = 1$ or $\lambda = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}$, the solution of these equations gives six possible sets of values of the form

$$a = \tfrac{1}{3}(g_1 + g_2 + g_3), \qquad (9)$$

$$b = \tfrac{1}{3}(g_1 + \lambda g_2 + \lambda^2 g_3), \qquad (10)$$

$$c = \tfrac{1}{3}(g_1 + \lambda^2 g_2 + \lambda g_3). \qquad (11)$$

The permutation of $g_1, g_2, g_3$ gives the other five sets. These six sets are all the solutions possible, since equation (7) is of the second degree and (8) of the third.

Substituting these values, we get

$$\phi = \tfrac{1}{3}g_1(1 + \iota + \iota^2) + \tfrac{1}{3}g_2(1 + \lambda\iota + \lambda^2\iota^2) + \tfrac{1}{3}g_3(1 + \lambda^2\iota + \lambda\iota^2). \qquad (12)$$

Let

$$x_1 = \tfrac{1}{3}(1 + \iota + \iota^2), \qquad (13)$$

$$x_2 = \tfrac{1}{3}(1 + \lambda\iota + \lambda^2\iota^2), \qquad (14)$$

$$x_3 = \tfrac{1}{3}(1 + \lambda^2\iota + \lambda\iota^2). \qquad (15)$$

We derive from these immediately

$$\left.\begin{aligned} x_1^2 &= x_1, & x_1x_2 &= 0 = x_2x_1, \\ x_2^2 &= x_2, & x_1x_3 &= 0 = x_3x_1, \\ x_3^2 &= x_3, & x_2x_3 &= 0 = x_3x_2, \end{aligned}\right\} \qquad (16)$$

$$\phi = g_1x_1 + g_2x_2 + g_3x_3. \qquad (17)$$

Now if $\rho_1, \rho_2, \rho_3$ are the axes of $\phi$, i. e. solutions of the equation $V.\rho\phi\rho = 0$, we can write $\phi$ in the form

$$\phi = \frac{g_1\rho_1 S.\rho_2\rho_3(\,) + g_2\rho_2 S.\rho_3\rho_1(\,) + g_3\rho_3 S.\rho_1\rho_2(\,)}{S.\rho_1\rho_2\rho_3}. \qquad (18)$$

Operating by $\varkappa_1$, we get

$$\varkappa_1\phi = g_1\varkappa_1 = \frac{g_1\varkappa_1\rho_1 S\,.\,\rho_2\rho_3\,(\;) + g_2\varkappa_1\rho_2 S\rho_3\rho_1\,(\;) + g_3\varkappa_1\rho_3 S\rho_1\rho_2\,(\;)}{S\,.\,\rho_1\rho_2\rho_3}\,. \tag{19}$$

Operating by this operator on $\rho = x\rho_1 + y\rho_2 + z\rho_3$, where $\rho$ is *any* vector, we get

$$xg_1\varkappa_1\rho_1 + yg_1\varkappa_1\rho_2 + zg_1\varkappa_1\rho_3 = xg_1\varkappa_1\rho_1 + yg_2\varkappa_1\rho_2 + zg_3\varkappa_1\rho_3\,.$$

Hence, since $x$, $y$, $z$ are independent,

$$g_1\varkappa_1\rho_2 = g_2\varkappa_1\rho_2\,,\quad g_1\varkappa_1\rho_3 = g_3\varkappa_1\rho_3\,.$$

These equations are possible generally only if

$$\varkappa_1\rho_2 = 0\,,\quad \varkappa_1\rho_3 = 0\,,$$

$$\therefore\ \varkappa_1 = \frac{\varkappa_1\rho_1 S\,.\,\rho_2\rho_3\,(\;)}{S\,.\,\rho_1\rho_2\rho_3}\,, \tag{20}$$

$$\varkappa_2 = \frac{\varkappa_2\rho_2 S\,.\,\rho_3\rho_1\,(\;)}{S\,.\,\rho_1\rho_2\rho_3}\,, \tag{21}$$

$$\varkappa_3 = \frac{\varkappa_3\rho_3 S\,.\,\rho_1\rho_2\,(\;)}{S\,.\,\rho_1\rho_2\rho_3}\,. \tag{22}$$

Let $\varkappa_1\rho_1 = x\rho_1 + y\rho_2 + z\rho_3$.

Operating by $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ in turn we obtain

$$\begin{aligned} \varkappa_1\rho_1 &= x\varkappa_1\rho_1\,, & \therefore\ x &= 1\,. \\ y\varkappa_2\rho_2 &= 0\,, & \therefore\ y &= 0\,. \\ z\varkappa_3\rho_3 &= 0\,, & \therefore\ z &= 0\,. \end{aligned}$$

Treating each $\varkappa$ in the same manner and substituting in (20), (21), (22), we have

$$\left.\begin{aligned} \varkappa_1 &= \frac{\rho_1 S\rho_2\rho_3\,(\;)}{S\,.\,\rho_1\rho_2\rho_3}\,, \\ \varkappa_2 &= \frac{\rho_2 S\rho_3\rho_1\,(\;)}{S\,.\,\rho_1\rho_2\rho_3}\,, \\ \varkappa_3 &= \frac{\rho_3 S\rho_1\rho_2\,(\;)}{S\,.\,\rho_1\rho_2\rho_3}\,. \end{aligned}\right\} \tag{23}$$

Therefore, finally,

$$\left.\begin{aligned} \iota &= \frac{\rho_1 S\rho_2\rho_3\,(\;) + \lambda^2\rho_2 S\rho_3\rho_1\,(\;) + \lambda\rho_3 S.\rho_1\rho_2\,(\;)}{S.\rho_1\rho_2\rho_3}, \\ \iota^2 &= \frac{\rho_1 S\rho_2\rho_3\,(\;) + \lambda\rho_2 S\rho_3\rho_1\,(\;) + \lambda^2\rho_3 S.\rho_1\rho_2\,(\;)}{S.\rho_1\rho_2\rho_3}, \\ \iota^3 = 1 &= \frac{\rho_1 S\rho_2\rho_3\,(\;) + \rho_2 S\rho_3\rho_1\,(\;) + \rho_3 S.\rho_1\rho_2\,(\;)}{S.\rho_1\rho_2\rho_3}. \end{aligned}\right\} \qquad (24)$$

This completes the solution, as we now have $a$, $b$, $c$ expressed in terms of the roots and $\iota$ in terms of the axes. [We observe further that the six solutions mentioned are in the end one solution, as if we interchange $g_2$ and $g_3$ in (9–11) we likewise interchange $\rho_2$ and $\rho_3$ in (24).]

If the axes are not given, we may determine the $x$'s and $\iota$ directly from the roots and $\phi$, or the numbers $a$, $b$, $c$ and $\phi$, thus:

$$\left.\begin{aligned} 1 &= x_1 + x_2 + x_3, \\ \phi &= g_1x_1 + g_2x_2 + g_3x_3, \\ \phi^2 &= g_1^2x_1 + g_2^2x_2 + g_3^2x_3. \end{aligned}\right\} \qquad (25)$$

Hence, solving for $x_1$, $x_2$, $x_3$,

$$\left.\begin{aligned} x_1 &= \frac{(\phi - g_2)(\phi - g_3)}{(g_1 - g_2)(g_1 - g_3)}, \\ x_2 &= \frac{(\phi - g_3)(\phi - g_1)}{(g_2 - g_3)(g_2 - g_1)}, \\ x_3 &= \frac{(\phi - g_1)(\phi - g_2)}{(g_1 - g_2)(g_1 - g_3)}. \end{aligned}\right\} \qquad (26)$$

These values may be substituted in the values of $\iota$ and $\iota^2$ in terms of $x_1$, $x_2$, $x_3$. But we may also get $\iota$ and $\iota^2$ thus:

$$\phi - a = b\iota + c\iota^2,$$

$$\phi^2 - a^2 - 2bc = (c^2 + 2ab)\,\iota + (b^2 + 2ac)\,\iota^2,$$

hence

$$\iota = \frac{c\phi^2 - (b^2 + 2ac)\,\phi + ab^2 + ca^2 - 2bc^2}{c^3 - b^3}, \qquad (27)$$

$$\iota^2 = \frac{b\phi^2 - (c^2 + 2ab)\,\phi + ac^2 + ba^2 - 2b^2c}{b^3 - c^3}, \qquad (28)$$

These in turn will give $x_1, x_2, x_3$ in terms of $a$, $b$, $c$ and $\phi$.

2. If two operators are equal they must have the same $a$, $b$, $c$ and $\iota$.

For if
$$\left.\begin{aligned}\psi &= a' + b'\iota' + c'\iota'^2,\\ = \phi &= a + b\iota + c\iota^2,\end{aligned}\right\} \tag{29}$$
we have at once $a = a'$;
$$\therefore\; b\iota + c\iota^2 = b'\iota' + c'\iota'^2, \tag{30}$$
$$\therefore\; b\iota^2 + c = b'\iota\iota' + c'\iota\iota'^2, \tag{31}$$
$$= b'\frac{\rho_1 S\rho_2\rho_3\,\iota'(\;) + \lambda^2\rho_2 S\rho_3\rho_1\,\iota'(\;) + \lambda\rho_3 S\rho_1\rho_2\,\iota'(\;)}{S.\rho_1\rho_2\rho_3}$$
$$+ c'\frac{\rho_1 S\rho_2\rho_3\,\iota'^2(\;) + \lambda^2\rho_2 S\rho_3\rho_1\,\iota'^2(\;) + \lambda\rho_3 S.\rho_1\rho_2\,\iota'^2(\;)}{S.\rho_1\rho_2\rho_3}. \tag{32}$$
Let the axes of $\psi$ be $\rho_1', \rho_2', \rho_3'$. Then
$$b\iota^2\rho_1' + c\rho_1' = b'\frac{\rho_1 S\rho_2\rho_3\rho_1'}{S\rho_1\rho_2\rho_3} + c'\frac{\rho_1 S\rho_2\rho_3\rho_1'}{S.\rho_1\rho_2\rho_3}. \tag{33}$$
Substituting for $\iota^2$ its value in (24),
$$\left.\begin{aligned}b + c &= b' + c',\\ b\lambda\, S\rho_3\rho_1\rho_1' + cS\rho_3\rho_1\rho_1' &= 0,\\ b\lambda^2 S\rho_1\rho_2\rho_1' + cS\rho_1\rho_2\rho_1' &= 0,\end{aligned}\right\} \tag{34}$$
$$\therefore\; S.\rho_3\rho_1\rho_1' = 0,\;\; S\rho_1\rho_2\rho_1' = 0. \tag{35}$$

Finally, we must have then
$$\left.\begin{aligned}\rho_1' &= \rho_1,\\ \text{and}\;\; \rho_2' &= \rho_2,\\ \rho_3' &= \rho_3,\end{aligned}\right\} \tag{36}$$
since tensors are not concerned.

Therefore $\iota = \iota'$ and $c = c'$, $b = b'$. (37)

But it is to be observed that since $(\iota^2)^2 = \iota$, and $(\iota^2)^3 = 1$, we cannot, knowing that
$$\psi = a' + b'\iota' + c'\iota'^2,$$
$$\phi = a + b\iota + c\iota^2,$$
$$\psi = \phi,$$

assume that $b'\iota' = b\iota$, $c'\iota'^2 = c'\iota^2$; it is necessary to find the axes, and writing the forms (24) identify $\iota'$ with $\iota$ or $\iota^2$ accordingly. (See Taber, "On Certain Identities in the Theory of Matrices," Amer. Jour., XIII, p. 165.)

3. The expression

$$bc\psi^2 + (b^3 + c^3 - 2abc)\,\psi + b^2c^2 - ab^3 - ac^3 + a^2bc$$

may be called the Hessian of the cubic in $\phi$. Its factors are

$$\left(\psi + \frac{c^2 - ab}{b}\right),\quad \left(\psi + \frac{b^2 - ac}{c}\right).$$

When its axes are properly chosen it plays a part in the theory of the triangle.

The expression

$$\left(\theta + \frac{b^2 + c^2 - ab - ac}{b + c}\right)\left(\theta + \frac{\lambda b^2 + \lambda^2 c^2 - \lambda ac - \lambda^2 ab}{\lambda^2 b + \lambda c}\right) \times\left(\theta + \frac{\lambda^2 b^2 + \lambda c^2 - \lambda^2 ac - \lambda ab}{\lambda b + \lambda^2 c}\right)$$

may be called the cubicovariant of the cubic in $\phi$. It also is of importance in the theory of the triangle.

The two invariants of the cubic in $\phi$ are

$$\left.\begin{aligned} H &= -bc = \tfrac{1}{9}(3m_1 - m_2^2), \\ G &= -(b^3 + c^3) = \tfrac{1}{27}(9m_1m_2 - 2m_2^3 - 27m). \end{aligned}\right\} \qquad (38)$$

The discriminant is

$$\Delta = (b^3 - c^3)^2 = \tfrac{1}{27}(27m^2 - 18mm_1m_2 + 4m_1^3 + 4mm_2^3 - m_1^2m_2^2). \qquad (39)$$

When this is negative the roots $g_1, g_2, g_3$ are all real; when it is positive, two roots are imaginary; when it is zero, there are two equal roots; when $G = 0$ and $H = 0$ there are three equal roots.

4. The vector function

$$\phi_1 = \phi - a = b\iota + c\iota^2$$

has for its cubic

$$\phi_1^3 - 3bc\phi_1 - (b^3 + c^3) = 0. \qquad (40)$$

Its roots are

$$\left.\begin{aligned} e_1 &= \tfrac{1}{3}(2g_1 - g_2 - g_3) = b + c, \\ e_2 &= \tfrac{1}{3}(2g_2 - g_3 - g_1) = \lambda^2 b + c\lambda, \\ e_3 &= \tfrac{1}{3}(2g_3 - g_1 - g_2) = \lambda b + \lambda^2 c, \\ & e_1 + e_2 + e_3 = 0. \end{aligned}\right\} \qquad (41)$$

Hence

$$b\iota + c\iota^2 = e_1x_1 + e_2x_2 + e_3x_3. \qquad (42)$$

Let

$$\rho\,.\,u$$

represent Weierstrass' $\wp$ function for the invariants

$$\left.\begin{aligned} g_2 &= 12bc = \tfrac{4}{3}(m_2^2 - 3m_1), \\ g_3 &= 4(b^3 + c^3) = \tfrac{4}{27}(27m + 2m_2^3 - 9m_1m_2). \end{aligned}\right\} \tag{43}$$

Then the periods satisfy the equations

$$\left.\begin{aligned} \wp\,.\,\omega_1 = e_1,\ \ \wp\,.\,\omega_2 = e_2,\ \ \wp\,.\,\omega_3 = e_3, \\ \omega_1 + \omega_2 + \omega_3 = 0. \end{aligned}\right\} \tag{44}$$

Now we may let

$$b\iota + c\iota^2 = \wp\,.\,\psi, \tag{45}$$

whence

$$\psi = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3. \tag{46}$$

Further,

$$\begin{aligned} (b\iota + c\iota^2) - e_\mu &= -\frac{\sigma(\psi + e_\mu)\sigma(\psi - e_\mu)}{\sigma^2\psi\,\sigma^2 e_\mu} = \left(\frac{\sigma_\mu\psi}{\sigma\psi}\right)^2 \\ &= \frac{e_\nu - e_\mu}{\mathrm{sn}^2\left(\sqrt{e_\nu - e_\mu}\,\psi,\ \sqrt{\dfrac{e_\nu - e_\mu}{e_{\nu'} - e_\mu}}\right)}. \end{aligned} \tag{47}$$

These results, startling at first sight, are useful in the geometry of the triangle.

5. Let the transverse of $\phi$ be $\phi'$. We have at once, since the roots are the same, that $\phi'$ has the same $a$, $b$, $c$ as $\phi$. But $\iota$ becomes

$$\iota' = \frac{V\rho_2\rho_3 S.\rho_1(\,) + \lambda^2 V\rho_3\rho_1 S.\rho_2(\,) + \lambda V\rho_1\rho_2 S\rho_3(\,)}{S.\rho_1\rho_2\rho_3}. \tag{48}$$

We have easily

$$\phi - \phi' = 2V.\varepsilon(\,) = \frac{g_1\rho_1 S.V\rho_2\rho_3(\,) - g_1 V\rho_2\rho_3 S\rho_1(\,) + \ldots\ldots}{S.\rho_1\rho_2\rho_3} \tag{49}$$

$$= \frac{g_1 V.(V\rho_1 V\rho_2\rho_3)(\,) + g_2 V.(V.\rho_2 V\rho_3\rho_1)(\,) + g_3 V.(V.\rho_3 V\rho_1\rho_2)(\,)}{S.\rho_1\rho_2\rho_3}, \tag{50}$$

$$\therefore\ 2\varepsilon = \frac{g_1 V\rho_1 V\rho_2\rho_3 + g_2 V\rho_2 V\rho_3\rho_1 + g_3 V\rho_3 V\rho_1\rho_2}{S.\rho_1\rho_2\rho_3} \tag{51}$$

$$= \frac{(g_2 - g_3)\rho_1 S\rho_2\rho_3 + (g_3 - g_1)\rho_2 S.\rho_3\rho_1 + (g_1 - g_2)\rho_3 S\rho_1\rho_2}{S.\rho_1\rho_2\rho_3} \tag{52}$$

$$= \frac{(e_2 - e_3)\rho_1 S.\rho_2\rho_3 + (e_3 - e_1)\rho_2 S\rho_3\rho_1 + (e_1 - e_2)\rho_3 S\rho_1\rho_2}{S.\rho_1\rho_2\rho_3}. \tag{53}$$

6. Let the tensor of $\phi$ be defined as the arithmetical cube root of $m$, i. e.

$$T.\phi = \sqrt[3]{[a^3 + b^3 + c^3 - 3abc]} = \sqrt[3]{m} = \sqrt[3]{g_1g_2g_3}. \tag{54}$$

Let the versor of $\phi$ be $U.\phi$,

$$U.\phi = \frac{a}{T\phi} + \frac{b}{T\phi}\iota + \frac{c}{T\phi}\iota^2 = \frac{\phi}{T\phi} \tag{55}$$

$$= \frac{g_1}{\sqrt[3]{g_1g_2g_3}}\varkappa_1 + \frac{g_2}{\sqrt[3]{g_1g_2g_3}}\varkappa_2 + \frac{g_3}{\sqrt[3]{g_1g_2g_3}}\varkappa_3. \tag{56}$$

Let

$$\left.\begin{aligned} \frac{g_1}{\sqrt[3]{g_1g_2g_3}} &= e^{\theta+\eta}, \\ \frac{g_2}{\sqrt[3]{g_1g_2g_3}} &= e^{\lambda\theta+\lambda^2\eta}, \\ \frac{g_3}{\sqrt[3]{g_1g_2g_3}} &= e^{\lambda^2\theta+\lambda\eta}, \end{aligned}\right\} \tag{57}$$

whence

$$\left.\begin{aligned} \therefore\ \frac{a}{T.\phi} &= \tfrac{1}{3}\left[e^{\theta+\eta} + e^{\lambda\theta+\lambda^2\eta} + e^{\lambda^2\theta+\lambda\eta}\right] = f_0(\theta, \eta), \\ \frac{b}{T.\phi} &= \tfrac{1}{3}\left[e^{\theta+\eta} + \lambda e^{\lambda\theta+\lambda^2\eta} + \lambda^2 e^{\lambda^2\theta+\lambda\eta}\right] = f_1(\theta, \eta), \\ \frac{c}{T.\phi} &= \tfrac{1}{3}\left[e^{\theta+\eta} + \lambda^2 e^{\lambda\theta+\lambda^2\eta} + \lambda e^{\lambda^2\theta+\lambda\eta}\right] = f_2(\theta, \eta), \end{aligned}\right\} \tag{58}$$

(compare Taber, Amer. Jour., XIII, p. 169 et seq.)

$$\therefore\ \phi = T\phi\left[f_0(\theta, \eta) + \iota f_1(\theta, \eta) + \iota^2 f_2(\theta, \eta)\right]. \tag{59}$$

If

$$\left.\begin{aligned} f_0\theta &= \tfrac{1}{3}(e^\theta + e^{\lambda\theta} + e^{\lambda^2\theta}), \\ f_1\theta &= \tfrac{1}{3}(e^\theta + \lambda e^{\lambda\theta} + \lambda^2 e^{\lambda^2\theta}), \\ f_2\theta &= \tfrac{1}{3}(e^\theta + \lambda^2 e^{\lambda\theta} + \lambda e^{\lambda^2\theta}), \end{aligned}\right\} \tag{60}$$

$$\phi = T\phi\,.\,[f_0\theta + \iota f_1\theta + \iota^2 f_2\theta][f_0\eta + \iota f_2\eta + \iota^2 f_1\eta]. \tag{61}$$

We see easily

$$\left.\begin{aligned} f_0\theta &= L_{n=\infty}\left\{1 + \frac{\theta^3}{3\,!} + \frac{\theta^6}{6\,!} + \ldots \frac{\theta^{3n}}{(3n)\,!}\right\}, \\ f_1\theta &= L_{n=\infty}\left\{\frac{\theta^2}{2\,!} + \frac{\theta^5}{5\,!} + \frac{\theta^8}{8\,!} + \ldots \frac{\theta^{3n-1}}{(3n-1)\,!}\right\}, \\ f_2\theta &= L_{n=\infty}\left\{\frac{\theta}{1\,!} + \frac{\theta^4}{4\,!} + \frac{\theta^7}{7\,!} + \ldots \frac{\theta^{3n-2}}{(3n-2)\,!}\right\}, \end{aligned}\right\} \tag{62}$$

and

$$\left.\begin{aligned} f_0\theta &= \tfrac{1}{3}\left(e^\theta + 2e^{-\frac{1}{2}\theta}\cos.\frac{\theta\sqrt{3}}{2}\right) , \\ f_1\theta &= \tfrac{1}{3}\left(e^\theta - e^{-\frac{1}{2}\theta}\cos\frac{\theta\sqrt{3}}{2} - \sqrt{3}\,.\,e^{-\frac{1}{2}\theta}\sin\frac{\theta\sqrt{3}}{2}\right), \\ f_2\theta &= \tfrac{1}{3}\left(e^\theta - e^{-\frac{1}{2}\theta}\cos\frac{\theta\sqrt{3}}{2} + \sqrt{3}\,.\,e^{-\frac{1}{2}\theta}\sin\frac{\theta\sqrt{3}}{2}\right). \end{aligned}\right\} \tag{63}$$

Also

$$\left.\begin{aligned} \theta + \eta &= \tfrac{2}{3}\lg g_1 - \tfrac{1}{3}\lg g_2 - \tfrac{1}{3}\lg g_3, \\ \lambda\theta + \lambda^2\eta &= -\tfrac{1}{3}\lg g_1 + \tfrac{2}{3}\lg g_2 - \tfrac{1}{3}\lg g_3, \end{aligned}\right\} \tag{64}$$

$$\left.\begin{aligned} \therefore\ \theta &= \tfrac{1}{3}\lg g_1 + \tfrac{1}{3}\lambda^2\lg g_2 + \tfrac{1}{3}\lambda\ \lg g_3, \\ \eta &= \tfrac{1}{3}\lg g_1 + \tfrac{1}{3}\lambda\ \lg g_2 + \tfrac{1}{3}\lambda^2\lg g_3, \end{aligned}\right\} \tag{65}$$

$$\left.\begin{aligned} \theta &= \tfrac{1}{3}\lg\left(g_1 g_2^{-\frac{1}{2}} g_3^{-\frac{1}{2}}\right) + \frac{\sqrt{-1}}{\sqrt{3}}\lg\left(g_2^{-\frac{1}{2}} g_3^{+\frac{1}{2}}\right), \\ \eta &= \tfrac{1}{3}\lg\left(g_1 g_2^{-\frac{1}{2}} g_3^{-\frac{1}{2}}\right) - \frac{\sqrt{-1}}{\sqrt{3}}\lg\left(g_2^{-\frac{1}{2}} g_3^{\frac{1}{2}}\right) , \end{aligned}\right\} \tag{66}$$

$$\left.\begin{aligned} \theta + \eta &= \tfrac{1}{3}\lg.\frac{g_1^3}{m}, \\ \theta - \eta &= \frac{\sqrt{-1}}{\sqrt{3}}\lg.\frac{g_3}{g_2}. \end{aligned}\right\} \tag{67}$$

7. Returning now to the solution given in §1, it was stated that the roots $g_1$, $g_2$, $g_3$ might be permuted, giving six different solutions of the equations there given. These six solutions suggest six related functions which I designate as conjugates and co-functions. Of course $a$ is the same for all.

The first of these six is derived by interchanging $g_2$ and $g_3$ (but leaving the axes unchanged); whence, since this interchanges $b$ and $c$,

$$\text{Co}.\,\phi = a + c\iota + b\iota^2. \tag{68}$$

We have

$$\phi + \text{Co}.\,\phi = 2a + (b + c)(\iota + \iota^2). \tag{69}$$

In this expression $\iota + \iota^2$ is an operator whose form is

$$\iota + \iota^2 = \frac{2\rho_1 S\rho_2\rho_3(\;) - \rho_2 S\rho_3\rho_1(\;) - \rho_3 S\rho_1\rho_2(\;)}{S.\rho_1\rho_2\rho_3}. \tag{70}$$

Hence its roots are 2, $-1$, $-1$, and axes $\rho_1$, $\rho_2$, $\rho_3$. It is of much importance in the theory of the triangle.

$$\phi - \text{Co}.\,\phi = (b - c)(\iota - \iota^2), \tag{71}$$

wherein
$$\iota - \iota^2 = \sqrt{-3}\,\frac{\rho_3 S\rho_1\rho_2\,() - \rho_2 S\,.\,\rho_3\rho_1\,()}{S\,.\,\rho_1\rho_2\rho_3}, \tag{72}$$
$$= -\sqrt{-3}\,\frac{(\rho_3 S\rho_2 + \rho_2 S\rho_3)\,V.\,\rho_1\,()}{S\,.\,\rho_1\rho_2\rho_3}. \tag{73}$$

Hence its roots are $\pm\sqrt{-3}$, and axes $\rho_3$ and $\rho_2$. Having but two axes its nullity is one, hence vacuity must also be at least one, and since the extension annulled is that along $\rho_1$, which is not an axis, its vacuity is just one. (Taber, Amer. Jour., XII, p. 365.)

When $$\phi = \text{Co.}\,\phi, \quad b = c \text{ and } g_2 = g_3. \tag{74}$$
Obviously
$$\text{Co.}\,\phi = T\phi\,[f_0\theta + f_1\theta\,.\,\iota^2 + f_2\theta\,.\,\iota][f_0\eta + \iota f_1\eta + \iota^2 f_2\eta]. \tag{75}$$

8. The next of the six is got by writing $g_3$, $g_1$, $g_2$ for $g_1$, $g_2$, $g_3$ respectively. It is the first conjugate of $\phi$:

$$\varkappa\,.\,\phi = a + \lambda b\iota + \lambda^2 c\iota^2, \tag{76}$$
$$= g_3\varkappa_1 + g_1\varkappa_2 + g_2\varkappa_3. \tag{77}$$

The second conjugate is $\varkappa\,.\,\varkappa\phi = \varkappa^2\phi$,

$$\varkappa^2.\,\phi = a + \lambda^2 b\iota + \lambda c\iota^2, \tag{78}$$
$$= g_2\varkappa_1 + g_3\varkappa_2 + g_1\varkappa_3. \tag{79}$$

The properties of these two are indicated by Taber, Amer. Jour., XIII, as also of the parts of each, of $\phi$, $\varkappa\phi$, $\varkappa^2\phi$, viz. $S\phi$, $V_1\phi$, $V_2\phi$.

9. The co-functions of the conjugates have of course the same relation to the conjugates as the co-function of $\phi$ has to $\phi$. Thus

$$\text{Co.}\,\varkappa\,.\,\phi = a + \lambda c\iota + \lambda^2 b\iota^2 \tag{80}$$
$$= \varkappa\,.\,\text{Co.}\,\phi, \tag{81}$$
$$\text{Co.}\,\varkappa^2.\,\phi = \varkappa^2.\,\text{Co.}\,\phi. \tag{82}$$

The six operators
$$\phi,\ \text{Co.}\,\phi,\ \varkappa\phi,\ \varkappa\,.\,\text{Co.}\,\phi,\ \varkappa^2\phi,\ \varkappa^2.\,\text{Co.}\,\phi$$
are the six related operators referred to in §7.

10. Various combinations of these produce coefficients of the forms

$$a + b \quad + c \quad ,$$
$$a + \lambda b \; + \lambda^2 c,$$
$$a + \lambda^2 b + \lambda c\;;$$
$$a^2 + b^2 + c^2,\;\; a^2 + \lambda b^2 + \lambda^2 c^2,\;\; a^2 + \lambda^2 b^2 + \lambda c^2;$$

etc., which might be called *triskew* symmetric.

## II.—*The Nonion Form of* $\phi$.

1. Let there be any three vectors chosen, not coplanar, say $\alpha$, $\beta$, $\gamma$. Then we may write

$$\phi\alpha = \frac{\alpha S\beta\gamma\phi\alpha + \beta S\gamma\alpha\phi\alpha + \gamma S\alpha\beta\phi\alpha}{S\alpha\beta\gamma}, \tag{1}$$

etc. If we write

$$\phi\alpha = \alpha_1,\;\; \phi\beta = \beta_1,\;\; \phi\gamma = \gamma_1, \tag{2}$$

since any vector $\rho$ may be expressed in terms of $\alpha$, $\beta$, $\gamma$, we have in the most general case

$$\begin{aligned}\phi = {} & \frac{\alpha S\beta\gamma\alpha_1 S.\beta\gamma\,(\,)}{S.\alpha\beta\gamma} + \frac{\beta S\gamma\alpha\alpha_1 S.\beta\gamma\,(\,)}{S.\alpha\beta\gamma} + \frac{\gamma S\alpha\beta\alpha_1 S.\beta\gamma\,(\,)}{S.\alpha\beta\gamma} \\ & + \frac{\alpha S\beta\gamma\beta_1 S.\gamma\alpha\,(\,)}{S.\alpha\beta\gamma} + \frac{\beta S\gamma\alpha\beta_1 S.\gamma\alpha\,(\,)}{S.\alpha\beta\gamma} + \frac{\gamma S\alpha\beta\beta_1 S.\gamma\alpha\,(\,)}{S.\alpha\beta\gamma} \\ & + \frac{\alpha S\beta\gamma\gamma_1 S.\alpha\beta\,(\,)}{S.\alpha\beta\gamma} + \frac{\beta S\gamma\alpha\gamma_1 S.\alpha\beta\,(\,)}{S.\alpha\beta\gamma} + \frac{\gamma S\alpha\beta\gamma_1 S.\alpha\beta\,(\,)}{S.\alpha\beta\gamma},\end{aligned} \tag{3}$$

$$\begin{aligned} & = \nu_1' S.\beta\gamma\alpha_1 + \nu_1'' S.\gamma\alpha\alpha_1 + \nu_1''' S.\alpha\beta\alpha_1 \\ & + \nu_2' S.\beta\gamma\beta_1 + \nu_2'' S.\gamma\alpha\beta_1 + \nu_2''' S.\alpha\beta\beta_1 \\ & + \nu_3' S.\beta\gamma\gamma_1 + \nu_3'' S.\gamma\alpha\gamma_1 + \nu_3''' S.\alpha\beta\gamma_1,\end{aligned} \tag{4}$$

wherein the form of the $\nu$'s is evident by comparison of (3) and (4).

The operators $\nu$ are such that

$$\nu_1' + \nu_2'' + \nu_3''' = 1, \tag{5}$$

$$\left.\begin{aligned}\nu_r^{(r)}\nu_r^{(r)} &= \nu_r^{(r)}, \\ \nu_r^{(s)}\nu_t^{(r)} &= \nu_t^{(s)}, \\ \nu_r^{(s)}\nu_y^{(t)} &= 0\;\;.\end{aligned}\right\} \tag{6}$$

They have the following multiplication-table:

| | $\nu_1'$ | $\nu_1''$ | $\nu_1'''$ | $\nu_2'$ | $\nu_2''$ | $\nu_2'''$ | $\nu_3'$ | $\nu_3''$ | $\nu_3'''$ |
|---|---|---|---|---|---|---|---|---|---|
| $\nu_1'$ | $\nu_1'$ | 0 | 0 | $\nu_2'$ | 0 | 0 | $\nu_3'$ | 0 | 0 |
| $\nu_1''$ | $\nu_1''$ | 0 | 0 | $\nu_2''$ | 0 | 0 | $\nu_3''$ | 0 | 0 |
| $\nu_1'''$ | $\nu_1'''$ | 0 | 0 | $\nu_2'''$ | 0 | 0 | $\nu_3'''$ | 0 | 0 |
| $\nu_2'$ | 0 | $\nu_1'$ | 0 | 0 | $\nu_2'$ | 0 | 0 | $\nu_3'$ | 0 |
| $\nu_2''$ | 0 | $\nu_1''$ | 0 | 0 | $\nu_2''$ | 0 | 0 | $\nu_3''$ | 0 |
| $\nu_2'''$ | 0 | $\nu_1'''$ | 0 | 0 | $\nu_2'''$ | 0 | 0 | $\nu_3'''$ | 0 |
| $\nu_3'$ | 0 | 0 | $\nu_1'$ | 0 | 0 | $\nu_2'$ | 0 | 0 | $\nu_3'$ |
| $\nu_3''$ | 0 | 0 | $\nu_1''$ | 0 | 0 | $\nu_2''$ | 0 | 0 | $\nu_3''$ |
| $\nu_3'''$ | 0 | 0 | $\nu_1'''$ | 0 | 0 | $\nu_2'''$ | 0 | 0 | $\nu_3'''$ |

2. Of these nine operators

$$\nu_1', \ \nu_2'', \ \nu_3'''$$

have each a cubic of the form

$$\nu^3 - \nu^2 = 0. \tag{7}$$

Both nullity and vacuity are two.

The remaining six have, since they each annul extension in two directions, a nullity two; and, since the axis is included in the extension annulled, vacuity three. The cubic is

$$\nu^3 = 0. \tag{8}$$

3. When $\alpha$, $\beta$, $\gamma$ coincide with the axes, all the terms of $\phi$ vanish except the three involving

$$\nu_1', \ \nu_2'', \ \nu_3'''.$$

The roots are in this case

$$S.\beta\gamma\alpha_1, \quad S.\gamma\alpha\beta_1, \quad S.\alpha\beta\gamma_1.$$

4. Let us write

$$\left.\begin{array}{lll} \nu_0 = \nu_1' + \nu_2'' + \nu_3''', & \nu_3 = \nu_1'' + \nu_2''' + \nu_3', & \nu_6 = \nu_1''' + \nu_2' + \nu_3'', \\ \nu_1 = \nu_1' + \lambda^2\nu_2'' + \lambda\nu_3''', & \nu_4 = \nu_1'' + \lambda^2\nu_2''' + \lambda\nu_3', & \nu_7 = \nu_1''' + \lambda^2\nu_2' + \lambda\nu_3'', \\ \nu_2 = \nu_1' + \lambda\nu_2'' + \lambda^2\nu_3''', & \nu_5 = \nu'' + \lambda\nu_2''' + \lambda^2\nu_3', & \nu_8 = \nu_1''' + \lambda\nu_2' + \lambda\nu_3''. \end{array}\right\} \tag{9}$$

Then it is easily verified that *these* $\nu$'s have the multiplication-table:

| | $\nu_0$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ | $\nu_7$ | $\nu_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $\nu_0$ | $\nu_0$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ | $\nu_7$ | $\nu_8$ |
| $\nu_1$ | $\nu_1$ | $\nu_2$ | $\nu_0$ | $\lambda^2\nu_4$ | $\lambda^2\nu_5$ | $\lambda^2\nu_3$ | $\lambda\nu_7$ | $\lambda\nu_8$ | $\lambda\nu_6$ |
| $\nu_2$ | $\nu_2$ | $\nu_0$ | $\nu_1$ | $\lambda\nu_5$ | $\lambda\nu_3$ | $\lambda\nu_4$ | $\lambda^2\nu_8$ | $\lambda^2\nu_6$ | $\lambda^2\nu_7$. |
| $\nu_3$ | $\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ | $\nu_7$ | $\nu_8$ | $\nu_0$ | $\nu_1$ | $\nu_2$ |
| $\nu_4$ | $\nu_4$ | $\nu_5$ | $\nu_3$ | $\lambda^2\nu_7$ | $\lambda^2\nu_8$ | $\lambda^2\nu_6$ | $\lambda\nu_1$ | $\lambda\nu_2$ | $\lambda\nu_0$ |
| $\nu_5$ | $\nu_5$ | $\nu_3$ | $\nu_4$ | $\lambda\nu_8$ | $\lambda\nu_6$ | $\lambda\nu_7$ | $\lambda^2\nu_2$ | $\lambda^2\nu_0$ | $\lambda^2\nu_1$ |
| $\nu_6$ | $\nu_6$ | $\nu_7$ | $\nu_8$ | $\nu_0$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | $\nu_5$ |
| $\nu_7$ | $\nu_7$ | $\nu_8$ | $\nu_6$ | $\lambda^2\nu_1$ | $\lambda^2\nu_2$ | $\lambda^2\nu_0$ | $\lambda\nu_4$ | $\lambda\nu_5$ | $\lambda\nu_3$ |
| $\nu_8$ | $\nu_8$ | $\nu_6$ | $\nu_7$ | $\lambda\nu_2$ | $\lambda\nu_0$ | $\lambda\nu_1$ | $\lambda^2\nu_5$ | $\lambda^2\nu_3$ | $\lambda^2\nu_4$ |

For each we have $\nu^3 = 1$. Also

$$\left.\begin{array}{c} \nu_3\nu_4 = \lambda\nu_4\nu_3 \\ \nu_4 = \lambda\nu_1\nu_3,\ \nu_5 = \lambda^2\nu_1^2\nu_3,\ \nu_6 = \nu_3^2,\ \nu_7 = \lambda^2\nu_1\nu_3^2,\ \nu_8 = \lambda\nu_1^2\nu_3^2, \\ \nu_0 = 1, \quad \nu_3\nu_1 = \lambda\nu_1\nu_3, \quad \nu_3^2\nu_1 = \lambda^2\nu_1\nu_3^2, \\ \nu_3\nu_1^2 = \lambda^2\nu_1^2\nu_3, \quad \nu_3^2\nu_1^2 = \lambda\nu_1^2\nu_3^2. \end{array}\right\} \tag{10}$$

5. Since each of these vids satisfies the equation

$$\nu^3 - 1 = 0, \tag{11}$$

it is a vid of the form $\iota$ of the first part of this paper. Writing them out in full, we have

$$\left.\begin{array}{l}
\nu_0 = \dfrac{\alpha S\beta\gamma\,(\,) + \beta S\gamma\alpha\,(\,) + \gamma S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_1 = \dfrac{\alpha S\beta\gamma\,(\,) + \lambda^2\beta S\gamma\alpha\,(\,) + \lambda\gamma S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_2 = \dfrac{\alpha S\beta\gamma\,(\,) + \lambda\beta S\gamma\alpha\,(\,) + \lambda^2\gamma S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_3 = \dfrac{\beta S\beta\gamma\,(\,) + \gamma S\gamma\alpha\,(\,) + \alpha S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_4 = \dfrac{\beta S\beta\gamma\,(\,) + \lambda^2\gamma S\gamma\alpha\,(\,) + \lambda\alpha S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_5 = \dfrac{\beta S\beta\gamma\,(\,) + \lambda\gamma S\gamma\alpha\,(\,) + \lambda^2\alpha S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_6 = \dfrac{\gamma S.\beta\gamma\,(\,) + \alpha S\gamma\alpha\,(\,) + \beta S.\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_7 = \dfrac{\gamma S\beta\gamma\,(\,) + \lambda^2\alpha S\gamma\alpha\,(\,) + \lambda\beta S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_8 = \dfrac{\gamma S\beta\gamma\,(\,) + \lambda\alpha S\gamma\alpha\,(\,) + \lambda^2\beta S.\alpha\beta\,(\,)}{S.\alpha\beta\gamma},
\end{array}\right\} \tag{12}$$

The axes and corresponding roots are:

$$\left.\begin{array}{llllllll}
1=\nu_0: & \alpha, & \beta, & \gamma; & 1, & 1, & 1; \\
\nu_1: & \alpha, & \beta, & \gamma; & 1, & \lambda^2, & \lambda; \\
\nu_2: & \alpha, & \beta, & \gamma; & 1, & \lambda, & \lambda^2; \\
\nu_3: & \alpha+\beta+\gamma, & \alpha+\lambda\beta+\lambda^2\gamma, & \alpha+\lambda^2\beta+\lambda\gamma; & 1, & \lambda^2, & \lambda; \\
\nu_4: & \alpha+\beta+\lambda^2\gamma, & \lambda^2\alpha+\beta+\gamma, & \alpha+\lambda^2\beta+\gamma; & 1, & \lambda^2, & \lambda; \\
\nu_5: & \lambda^2\alpha+\lambda^2\beta+\gamma, & \lambda^2\alpha+\beta+\lambda^2\gamma, & \alpha+\lambda^2\beta+\lambda^2\gamma; & 1, & \lambda^2, & \lambda; \\
\nu_6: & \alpha+\beta+\gamma, & \alpha+\lambda\beta+\lambda^2\gamma, & \alpha+\lambda^2\beta+\lambda\gamma; & 1, & \lambda, & \lambda^2; \\
\nu_7: & \lambda^2\alpha+\lambda^2\beta+\gamma, & \lambda^2\alpha+\beta+\lambda^2\gamma, & \alpha+\lambda^2\beta+\lambda^2\gamma; & \lambda^2, & 1, & \lambda; \\
\nu_8: & \alpha+\beta+\lambda^2\gamma, & \lambda^2\alpha+\beta+\gamma, & \alpha+\lambda^2\beta+\gamma; & \lambda, & \lambda^2, & 1.
\end{array}\right\} \quad (13)$$

6. Since $\alpha, \beta, \gamma$ may be taken in an infinity of ways, any vector operator $\phi$ may be broken up into nine elements in an infinite number of ways.

Since there are but nine nonion cube roots of unity, or nine solutions of the equation $\phi^3 = 1$, they are all expressed above, *for any particular set of vectors* $(\alpha, \beta, \gamma)$, with their axes and roots.

7. If the axes of $\phi$ are $\alpha, \beta, \gamma$, then

$$\phi = a_0 + a_1\nu_1 + a_2\nu_2. \quad (14)$$

Any other operator may be reduced to this nonion form in terms of $\alpha, \beta, \gamma$, giving

$$\psi = a_0' + \sum_1^8{}_r a_r'\nu_r. \quad (15)$$

Since the $\nu$'s are all unit operators, it is plain that

$$S.\psi = a_0'. \quad (16)$$

Again, if $\psi = \chi$, then the coefficients of any one of the $\nu$'s must be identically equal. For we can find a $\nu$, complementary to the one considered, whose product into it will give either $\nu_0$, $\lambda\nu_0$ or $\lambda^2\nu_0$. Then if we multiply each side by this operator and take scalars, we have at once the result stated.

8. We have

$$\left.\begin{array}{l}
\phi\psi = a_0a_0' + a_1a_2' + a_2a_1' + \nu_1(a_0a_1' + a_1a_0' + a_2a_2') + \nu_2(a_0a_2' + a_1a_1' + a_2a_0') \\
\quad + \nu_3(a_0a_3' + \lambda^2a_1a_5' + \lambda a_2a_4') + \nu_4(a_0a_4' + \lambda^2a_1a_3' + \lambda a_2a_5') \\
\quad + \nu_5(a_0a_5' + \lambda^2a_1a_4' + \lambda a_2a_3') + \nu_6(a_0a_6' + \lambda a_1a_8' + \lambda^2a_2a_7') \\
\quad + \nu_7(a_0a_7' + \lambda a_1a_6' + \lambda^2a_2a_8') + \nu_8(a_0a_8' + \lambda a_1a_7' + \lambda^2a_2a_6'),
\end{array}\right\} \quad (17)$$

$$\left.\begin{aligned}\psi\phi = a_0a'_0 + a_1a'_2 + a_2a'_1 + \nu_1(a_0a'_1 + a_1a'_0 + a_2a'_2) + \nu_2(a_0a'_2 + a_1a'_1 + a_2a'_0)\\ + \nu_3(a_0a'_3 + a_1a'_5 + a_2a'_4) + \nu_4(a_0a'_4 + a_1a'_3 + a_2a'_5) + \nu_5(a_0a'_5 + a_1a'_4 + a_2a'_3)\\ + \nu_6(a_0a'_6 + a_1a'_8 + a_2a'_7) + \nu_7(a_0a'_7 + a_1a'_6 + a_2a'_8) + \nu_8(a_0a'_8 + a_1a'_7 + a_2a'_6).\end{aligned}\right\} \quad (18)$$

It is easily apparent now that always

$$S.\phi\psi = S.\psi\phi, \quad S.\nu_1\phi\psi = S.\nu_1\psi\phi, \quad S.\nu_2\phi\psi = S.\nu_2\psi\phi. \quad (19)$$

Also

$$\begin{aligned}\phi\psi - \psi\phi = (\lambda - 1)\{\nu_3(-\lambda^2a_1a'_5 + a_2a'_4) + \nu_4(-\lambda^2a_1a'_3 + a_2a'_5)\\ + \nu_5(-\lambda^2a_1a'_4 + a_2a'_3) + \nu_6(a_1a'_8 - \lambda^2a_2a'_7) + \nu_7(a_1a'_6 - \lambda^2a_2a'_8)\\ + \nu_8(a_1a'_7 - \lambda^2a_2a'_6)\}.\end{aligned} \quad (20)$$

We deduce from this that $\phi\psi = \psi\phi$ only when each of these coefficients vanishes; that is, when either—

*First:*

$$\frac{a_1}{a_2} = \frac{\lambda a'_3}{a'_4} = \frac{\lambda a'_4}{a'_5} = \frac{\lambda a'_5}{a'_3} = \frac{\lambda^2 a'_6}{a'_7} = \frac{\lambda^2 a'_7}{a'_8} = \frac{\lambda^2 a'_8}{a'_6}, \quad (21)$$

i. e. when

$$\frac{a'_3}{a'_4} = \frac{a'_4}{a'_5} = \frac{a'_5}{a'_3},$$

whence

$$\left.\begin{aligned}&a'_4 = \lambda^m a'_3,\\ &a'_5 = \lambda^{2m}a'_3, \qquad a_1 = \lambda^{2m}a_2,\\ &a'_7 = \lambda^m a'_6, \qquad m = 1 \text{ or } 2 \text{ or } 3,\\ &a'_8 = \lambda^{2m}a'_6,\end{aligned}\right\} \quad (22)$$

or *Second*, when $a_1 : a_2 \neq \lambda^{2m}$, then $a'_3, a'_4, a'_5, a'_6, a'_7, a'_8$ are all zero.

[$a_1$, $a_2$, etc., may be imaginary, or complex, whence we cannot break these up farther.]

In the second case $\phi$ and $\psi$ are clearly *coaxal*. In the first case, if

$$\left.\begin{aligned}&m = 3 \text{ and } a_1 = a_2, \quad g_1 + \lambda^2g_2 + \lambda g_3 = g_1 + \lambda g_2 + \lambda^2g_3, \quad \therefore g_2 = g_3;\\ &m = 1, \quad a_1 = \lambda a_2, \quad g_1 + \lambda^2g_2 + \lambda g_3 = \lambda g_1 + \lambda^2g_2 + g_3, \quad \therefore g_1 = g_3;\\ &m = 2, \quad a_1 = \lambda^2a_2, \quad g_1 + \lambda^2g_2 + \lambda g_3 = \lambda^2g_1 + g_2 + \lambda g_3, \quad \therefore g_1 = g_2;\end{aligned}\right\} \quad (23)$$

hence in either case, $\phi$ has two equal roots.

Substituting the values of $a'_0 \ldots\ldots a'_8$ in (15) and reducing by (12),

$$\psi = \frac{(a'_0 + a'_1 + a'_2)\,\alpha S\beta\gamma(\,)}{S\alpha\beta\gamma} + \frac{(a'_0 + \lambda^2a'_1 + \lambda a'_2)\,\beta S\gamma\alpha(\,)}{S\alpha\beta\gamma} + \frac{(a'_0 + \lambda a'_1 + \lambda^2a'_2)\,\gamma S\alpha\beta(\,)}{S.\alpha\beta\gamma}$$

$$+\left.\begin{aligned}&\frac{3a'_3\,\beta S\beta\gamma(\,)}{S.\alpha\beta\gamma}\\ &+\frac{3a'_6\,\gamma S\beta\gamma(\,)}{S.\alpha\beta\gamma}\end{aligned}\right\} \text{ or } +\left.\begin{aligned}&\frac{3a'_3\,\gamma S\gamma\alpha(\,)}{S.\alpha\beta\gamma}\\ &+\frac{3a'_6\,\alpha S\gamma\alpha(\,)}{S.\alpha\beta\gamma}\end{aligned}\right\} \text{ or } +\left.\begin{aligned}&\frac{3a'_3\,\alpha S\alpha\beta(\,)}{S.\alpha\beta\gamma}\\ &+\frac{3a'_6\,\beta S\alpha\beta}{S.\alpha\beta\gamma}\end{aligned}\right\} \quad (24)$$

Therefore, $\psi$ must have for two of its axes, two directions that are taken in the plane of the axes of $\phi$ corresponding to the equal roots of $\phi$. Further, since we might have expressed $\phi$ in terms of $\psi$, and similar conclusions would result, hence $\psi$ must have equal roots, and we have finally:

Two operators $\phi$, $\psi$ are commutative only when (1) they have the same axes; (2) each has a pair of equal roots, the planes of the two pairs being one and the same plane. The product will be of the same nature as the factors.*

As a corollary, $\phi$ and $\phi'$ are never commutative unless $\phi = \phi'$.

In this case $\alpha \perp \beta \perp \gamma$ and the nine unit nonions become much simpler.

9. When $\phi$ and $\psi$ are commutative, then

$$\psi = a_0' + a_1'\nu_1 + a_2'\nu_2 + a_3'(\nu_3 + \lambda^m\nu_4 + \lambda^{2m}\nu_5) + a_6'(\nu_6 + \lambda^m\nu_7 + \lambda^{2m}\nu_8), \quad (23)$$

in which $a_3'$ or $a_6'$ or both may be zero.

10. There is a large field remaining here which is of great use in any application of Quaternions, aside from its importance as a part of the algebra Nonions.

ILLINOIS COLLEGE, *August* 1, 1895.

* For a fuller discussion see paper read before the American Mathematical Society in August, 1896.

# *On Certain Applications of the Theory of Probability to Physical Phenomena.*

BY G. H. BRYAN, SC. D., F. R. S.,
*Professor of Mathematics in the University College of North Wales.*

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Since the discovery of the Principle of Conservation of Energy, it has been one of the chief objects of Mathematical Physics to account for all physical phenomena as far as possible by means of the principles of Theoretical Dynamics. The main difficulty in doing so arises from the fact that the equations of motion of an ideal system always represent perfectly reversible motions, whereas in the phenomena of nature reversible processes are conspicuous by their absence. To account for the Second Law of Thermodynamics, as applied to irreversible transformations, something is needed besides the ordinary equations of dynamics, and the Theory of Probability suggests one possible way out of the difficulty.

According to the Kinetic Theory, the molecules of a gas are always moving about freely in all directions, and when they come within a certain distance apart, they repel each other and rebound in the same way that perfectly elastic bodies would do after a collision. As Lord Salisbury has said, "What the atom of each element is, whether it is a movement or a thing or a vortex or a point having inertia .... all these questions remain surrounded by a darkness as profound as ever." But whatever be the correct answer to these unsolved enigmas, it is remarkable that determinations of the ratios of the specific heats of gases accord on the whole very fairly closely with the theory according to which the molecules of gases can, so far as thermal phenomena are concerned, be represented by rigid bodies, whose energy is in most cases partly translational and rotational. The new gas Argon discovered by Lord Rayleigh and Professor Ramsay is peculiar in being the only gas besides mercury-vapor in which the energy is purely translational.

Corresponding to the property that the entropy of a system of unequally heated bodies always tends to a maximum, it has been established by Boltzmann that when a medium consisting of gas-molecules has been disturbed and is left to itself, a certain function tends to a minimum, and this minimum value is attained when the motions are distributed according to the well-known Boltzmann-Maxwell Law.

Now it has been objected, in the course of a controversy in *Nature*, that if the velocities of every molecule were all exactly reversed, the system would exactly retrace its steps, so that the function in question would increase instead of decreasing; and it has further been suggested that since the number of direct motions is necessarily equal to the number of reverse motions, the chances are just as great that the function increases as that it decreases. The fallacy of this argument lies in the fact that the probability of an event depends not only on the number of ways in which it can happen, but also quite as much on the number of ways in which it can fail. And if this is greater for the reversed than for the direct motion, we may safely infer that in the ordinary course of events there will be a preponderance of direct over reverse motions.

Thus if a number of bodies have been projected from a common point in different directions, it is theoretically possible to imagine their motions at any instant reversed, so that they all return to the point of projection, but if we were to attempt to throw them back we should find it practically impossible to bring them together again, because the chances of our failing would be so overwhelmingly great.

In the problem considered by Boltzmann, a number of molecules of gas are supposed to start with their motions distributed according to a law different to that which they have in their equilibrium state, such initial distribution being presumably the result of an artificially produced disturbance. For example, the gas may have been unequally heated, or two portions of it at different pressures may have been allowed to mix. Because the molecules tend to assume their equilibrium distribution, it is none the more probable that if left to themselves they will ever tend to return to their initial state, as the odds are overwhelmingly against their doing so as a matter of pure chance. In the direct motion the number of ways that the molecules can move is limited by the initial conditions, in the reversed motion it is limited by the final conditions. Hence, in order to prove that there is in general a tendency among the molecules of a gas to assume the Boltzmann-Maxwell distribution, it is sufficient to show that the number of ways in which the molecules can move consistently with this distribution is greater than the number of ways in which they could move if their motions were distributed in any other arbitrary manner. The following solution of this problem is due to Boltzmann, whose original investigations are, however, very lengthy.

CASE I.—Let there be a finite though large number ($n$) of molecules, and suppose that the kinetic energy of each molecule must have one or other of a discrete series of values $e$, $2e$, $3e$, .... $pe$. Taking the total energy $T$ of the system as equal to $\lambda e$, let us investigate the probability that it should be divided between the molecules in a given manner, each value of the energy being *a priori* equally probable for any single molecule. If $w_0$, $w_1$, $w_2$, .... $w_p$ be the numbers of molecules having energies $0$, $e$, $2e$, .... $pe$, the number of permutations of the molecules satisfying this distribution will be

$$P = \frac{n!}{w_0!\, w_1! \ldots . w_p!}, \qquad (1)$$

where $w_0$, $w_1$, $w_2$, .... $w_p$ are subject to the conditions

$$w_0 + w_1 + \ldots . + w_p = n, \qquad (2)$$

$$w_1 + 2w_2 + \ldots . + pw_p = \lambda. \qquad (3)$$

The most probable distribution is that in which the number of permutations is greatest. Let $H$ denote the logarithm of the denominator of $P$, that is, let

$$H = \log(w_0!) + \log(w_1!) + \ldots . \qquad (4)$$

We have therefore to make $H$ a minimum subject to the conditions (2), (3). Now when $n$ is very great, $w!$ may be replaced by its approximate value

$$\sqrt{(2\pi)}\left(\frac{w}{\varepsilon}\right)^w, \qquad (5)$$

where $\varepsilon$ is the base of the Napierian logarithms; therefore by substitution

$$H = \tfrac{1}{2}\log 2\pi + \sum_{\iota=0}^{\iota=p} w_\iota(\log w_\iota - 1).$$

Passing to the limiting case in which the energy is capable of continuous variation, let $f(x)\,dx$ denote the number of molecules whose energy is between $x$ and $x + dx$. Then we may put

$$w_0 = ef(0), \quad w_1 = ef(e) \ldots . w_p = ef(pe), \qquad (6)$$

where $e = dx$ in the limit. Hence the problem reduces to finding the minimum of

$$H = \int_0^\infty f(x)\{\log f(x) - 1\}\,dx, \qquad (7)$$

subject to the conditions

$$n = \int_0^\infty f(x)\,dx, \tag{8}$$

$$T = \int_0^\infty x f(x)\,dx. \tag{9}$$

The solution is

$$f(x)\,dx = Ce^{-hx}\,dx. \tag{10}$$

This therefore is the most probable distribution of the energy among the molecules on the hypothesis that all values of the energy are *a priori* equally probable for any one molecule. If $m$ is the mass of a molecule, the most probable number of molecules with velocities between $v$ and $v + dw$ is

$$Ce^{-\frac{1}{2}hmv^2} mv\,dv. \tag{11}$$

This is the Boltzmann-Maxwell distribution of speed for a system of monatomic molecules *moving in one plane.*

CASE II.—To obtain the Boltzmann-Maxwell distribution for molecules moving in three dimensions, it is necessary to make a different assumption with regard to the *a priori* probabilities for the individual molecules. We therefore assume that if $u$, $v$, $w$ be the velocity components of a molecule along the axes of coordinates, all values of $u$, $v$, $w$ are *a priori* equally probable.

The problem of determining the most probable distribution now reduces to finding the minimum of $H$ where

$$H = \int\int\int \{f \log f - 1\}\,du\,dv\,dw, \tag{12}$$

subject to the conditions

$$n = \int\int\int f\,du\,dv\,dw, \tag{13}$$

$$T = \tfrac{1}{2}m\int\int\int (u^2 + v^2 + w^2)\,du\,dv\,dw. \tag{14}$$

CASE III.—Let the gas consist of molecules moving about freely or in a field of external force, each molecule having $r$ degrees of freedom and its motion being specified by the generalized coordinates $q_1, q_2, \ldots. q_r$ and the corresponding generalized momenta, $p_1, p_2, \ldots. p_r$. Then it has been shown (see Watson, "Kinetic Theory of Gases") that throughout the motion of a molecule the multiple differential

$$dp_1 . dp_2 \ldots. dp_r . dq_1 . dq_2 \ldots. dq_r$$

is constant. We are therefore justified in assuming that the *a priori* probability that the coordinates and momenta of a molecule lie between the limits of this multiple differential is independent of the actual values of the coordinates and momenta. As Boltzmann puts it, we may suppose an urn filled with tickets, each ticket inscribed with a set of values of $p_1, p_2 \ldots\ldots q_r$, the number of tickets in which the values in question lie between the limits $dp_1 \ldots\ldots dq_r$ being measured by the product of the multiple differential $dp_1 \ldots\ldots dq_r$ into a constant. In the case of a mixture of gases, there would have to be a number of urns equal to the number of gases, and the number of tickets drawn from each urn would have to equal the number of molecules of the gas in question in the mixture. We suppose that a drawing from the various urns has been made, and that the energy of the assemblage of molecules represented by the selected tickets is known, and the problem is to find what is *a posteriori* the most probable distribution of the values of $p_1, p_2, \ldots\ldots q_r$ consistent with this energy condition. [The terms *a priori* and *a posteriori probability* are defined in any text-book on algebra.]

The final result is that if $f(p_1 \ldots\ldots q_r)\, dp_1 \ldots\ldots dq_r$ denote the number of molecules whose coordinates and momenta lie within the limits of the multiple differential, then the most probable distribution is found by making $H$ a minimum, where

$$H = \sum \int \{ f \log f - 1 \}\, dp_r \ldots\ldots dq_r, \tag{15}$$

and the sign of summation $\sum$ refers to the different kinds of molecules when a mixture of several gases is considered.

The expression $H$ is in every case *Boltzmann's minimum function.*

Case IV.—The particular assumption as to the law of *a priori* probability precludes the above investigations from furnishing a complete proof of the Boltzmann-Maxwell Law. In a subsequent paper Boltzmann has removed this restriction and has considered the *a posteriori* probabilities corresponding to any assumed law of *a priori* probability. In other words, we start with a large number ($N$) of molecules having a given distribution of energy, and from them a smaller number ($n$) are selected, and their mean energy is found to have a certain value which may be either the same or different from that of the original $N$. It is required to find the most probable law of distribution in the $n$ selected molecules, or generally the probability of any given distribution.

Boltzmann first considers the case where the original molecules follow the Boltzmann-Maxwell Law for two-dimensional space and points out the necessary modifications for space of three dimensions. In the general case, supposing $f_1, f_2, \ldots . f_p$ to denote the *a priori* probabilities of a molecule having energies $e, 2e, \ldots . pe$, the *a posteriori* probability of a combination in which the numbers of molecules having these energies are $w_0, w_1, \ldots . w_p$ is proportional to $\Omega$ where

$$\Omega = f_0^{w_0} f_1^{w_1} \ldots . f_p^{w_p} \frac{n!}{w_0!\, w_1!\, w_2! \ldots .}, \tag{16}$$

where as before

$$\sum w_i = n, \quad \sum i w_i = \lambda.$$

The approximate expression for $w!$ now gives

$$\log \Omega = \sum w_i \log f_i - \sum w_i \log w_i + \text{const.},$$

and Boltzmann finds the following results:

If the mean energy of the selected $n$ molecules is equal to the mean energy of the original $N$, the most probable distribution of energy in the latter is identical with the distribution in the former.

If, however, the mean energy of the smaller number is unequal to that of the larger, the most probable distribution is that given by the form

$$w_k = \frac{an}{N} f_k e^{hke}. \tag{17}$$

It thus appears that even the Theory of Probability does not furnish us with a *conclusive* proof of the Boltzmann-Maxwell Law. That the law in question represents accurately the state of the molecules in a perfect gas, and approximately their state in an ordinary gas, cannot be doubted, but directly we attempt to generalize the law by applying it to assemblages of densely crowded molecules we are confronted with the necessity of making some assumption or other, and the above treatment shows that even probability considerations do not afford a sure way out of the difficulty.

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# *On Three Septic Surfaces.*

By John E. Hill.



If, in the general cubo-cubic transformation between two spaces,* we cause the principal sextic of one space to degenerate into a twisted quintic of deficiency 2, $C'_5$, and into a right line, $C'_1$, meeting $C'_5$ twice, to the general cubic surface, $S'_3$, upon which $C'_1$ lies, there will correspond, in the second space, a septic surface, $S^{(7)}_{1^3, 5^2}$, upon which $C_1$ is triple and $C_5$ is double. If, however, the principal sextic of the first space breaks up into a twisted quartic of the second kind, $C'_4$, and into a conic, $C'_2$, meeting $C'_4$ four times, to the general cubic surface, $S'_3$, passed through $C'_2$, there will correspond, in the second space, a septic surface, $S^{(7)}_{2^3, 4^2}$, possessing $C_4$ doubly and $C_2$ triply. If, however, finally, the principal sextic of the first space degenerates completely, to the general cubic, $S'_3$, passed through the two transversals, $A'_1$, $B'_1$, and one line, $C'_1$, of the remaining ingredients, $C'_1$, $D'_1$, $E'_1$, $F'_1$, there will correspond, in the second space, a septic surface, $S^{(7)}_{1^3, 1^3, 1^3, 1^2, 1^2, 1^2}$, possessing $A_1$, $B_1$, $C_1$ triply and $D_1$, $E_1$, $F_1$ doubly. It is the purpose of this paper to study some of the properties of these three septic surfaces by means of their plane representations.

The general plane section of *the surface*, $S^{(7)}_{1^3, 1^3, 1^3, 1^2, 1^2, 1^2}$, is imaged, in its simplest plane representation, by a quartic curve passing through nine simple fundamental points; this surface, therefore, possesses only the nine lines whose images are these points and only the 36 conics whose images are those lines of the representative plane which join these points, two by two. Three of these conics, however, are the double lines possessed by the surface; for, if we number the fundamental points 1, 2, 3, 4, 5, 6, 7, 8, 9, the lines, say, 89, 97, 78 are the images of $A_1$, $B_1$ and $C_1$ respectively; so that the surface possesses, in fact, but 33 proper conics. There are 9 sheaves and 126 isolated examples of nodal

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* See the article, "On Quintic Surfaces," in the Mathematical Review, vol. 1, No. 1.

cubics upon the surface, imaged, respectively, by the 9 sheaves of lines centering at the fundamental points and by the conics determined by the fundamental points taken five by five; of these last cubics, however, those imaged by the conics 1 2 3 4 5, 1 2 3 4 6 and 5 6 7 8 9 are the triple lines, $A_1$, $B_1$ and $C_1$.

By close study of the relations existing between the images in the representative plane, we see that no two lines of the surface meet, while each line meets two conics each once, to which, consequently, it is coordinated. On the other hand, each conic meets two lines in such a way that, as mentioned above, to each line there is coordinated two conics. Two conics meet or do not meet according as they do not or do have a common coordinated line. There are, moreover, three pairs of conics in which the ingredients of each pair meet twice. To make this last fact clear, we must first consider the relations that exist between the simple and multiple lines, and the multiple lines and the conics of the surface.

Each double line is met by two and but two simple lines, and this in such a way that there are six of the nine simple lines of the surface that do not meet a double line. On the other hand, five simple lines meet each triple line and in such a way that there is no simple line that does not meet a triple line. Four simple lines are common to $A_1$ and $B_1$, one common to $A_1$ and $C_1$ and one common to $B_1$ and $C_1$. Consequently, of these nine simple lines, each one of six meets two triple lines while each of the remaining three meets one triple and two double lines. Since $A_1$ and $C_1$ determine a plane, the line common to both is the residue of the section of $S^{(7)}$ determined by that plane; and similarly in the case of $B_1$ and $C_1$.

Each double line is met once by 21 conics and not at all by the other 12. Of these 21, however, there are 6 that still meet a double line twice; for, since $A_1$ and $B_1$ each meet $D_1$, $E_1$ and $F_1$, there are six planes thus determined whose residue sections of $S^{(7)}$ are conics; two of these six conics meet $D_1$ twice, two meet $E_1$ twice and two meet $F_1$ twice and in such a way that no two of them are common to any two of the three lines, $D_1$, $E_1$, $F_1$; these three pairs of conics are imaged, respectively, by 67 and 57, 68 and 58, and 69 and 59. Each triple line is not met at all by 10 conics (counting the double lines), is met once by 20 and twice by six; and there is no conic that does not meet a triple line. Moreover, of the 33 proper conics, there are 12 that are met twice by a triple line while the remaining 21 are met once. This, therefore, leaves but 21 isolated planes, whose sections consist of conics and quintics.

Each of the 12 conics determines, with the triple line that meets it twice, a plane whose residue is still a conic. The 12 conics, therefore, must, apparently, group themselves in pairs and form with the triple lines six plane sections of $S^{(7)}$. In fact, however, we easily see, from the plane projection, that but three pairs are so formed, the remaining six conics each pairing with a double line to make up those sections with the triple lines that were mentioned in the last paragraph. So that, counting the double lines as conics, we see that 15 of the 36 conics group themselves in nine pairs (each double line entering as ingredient into two such pairs) in such a way that there are three of such planes through each triple line. Excluding the double lines, the 12 conics form three pairs and two triplets; each pair forms with $C_1$ a complete plane section, while neither ingredient of any pair meets $A_1$ nor $B_1$, but each meets each double line once. The two triplets are composed of conics that each, in the one case, meets $A_1$ twice and $B_1$ once and, in the other case, meets $A_1$ once and $B_1$ twice; not one of such conics meets $C_1$, but each meets one double line once.

The remaining 21 conics group themselves into an octuplet, a 12-tuplet and an isolated case. The octuplet is composed of conics that meet two triple and the three double lines each once and, further, divides itself into two quadruplets according as to whether $A_1$ or $B_1$ is the second triple line that is met. The 12-tuplet is distinguished by conics that meet the three triple lines and one double line each once and, consequently, still separates itself into three quadruplets according to which double line it is that is met. Altogether, therefore, the 33 conics group themselves into an isolated case, three pairs, two triplets and five quadruplets. The isolated case is that conic that meets $A_1$, $B_1$, $D_1$, $E_1$ and $F_1$ each once, without meeting $C_1$.

Of nodal cubics, which include both plane and twisted curves, we have already noted nine sheaves and 126 isolated cases (including the three triple lines). In each sheaf there are eight cubics that break up into a line and a conic; such are imaged by the lines of the sheaf that pass through any one of the remaining eight fundamental points. In three of such last cases, the conic becomes a double line and the planes are, in fact, identical with the planes of the sheaf of planes through that double line, being, really, planes in which the residue quintic degenerates into a quartic and a right line, two such degenerations taking place in each sheaf.

Every cubic of the nine sheaves is met once by one and but one line, except that in each sheaf there is one cubic that is met three times by such line, viz. the cubic of that plane in which the residue quartic breaks up into a cubic and a line. This will become apparent when such quartics are considered. Each cubic of these nine sheaves fails to meet eight conics while it meets each of the remainder once. No two cubics of the same sheaf meet, but each meets each cubic of the other sheaves once. In six of the nine sheaves, each cubic meets one triple line twice, each of the other two triple lines once and each double line once; in the remaining three sheaves, each cubic meets two triple lines twice, the third triple line once, one double line once and the remaining two double lines not at all. If we count the triple lines as isolated cubics, each cubic of the nine sheaves meets 70 isolated cubics once and the remainder twice.

Each isolated cubic meets five lines once while it fails to meet the other four. Including the double lines, it meets six conics twice, 20 conics once and ten not at all. It meets each cubic of four sheaves twice and each cubic of the remaining five sheaves once. It meets five of its kind thrice, 40 twice and 60 once, while it fails to meet the remaining 20. As will be noted below, in eleven cases the residue quartic breaks up into a triple line and a simple line. The relations of these cubics to the multiple lines of the surface will be more easily pointed out when we come to consider the complete sections.

There are no plane cubics other than those thus enumerated.

Plane quartics can only exist in three ways: (a) such as complete the sections of the planes of the above-noted cubics; (b) those that are the residues of planes having the triple lines as axes, and (c) such as are ingredients of degenerate cases of plane quintics and sextics. Of (a) there are 123 isolated cases imaged as conics through four fundamental points, their fifth points being determinate points, and nine sheaves imaged as non-singular cubics through eight fundamental points; of (b) there are three sheaves imaged the same as the 123 isolated cases of (a) except that their fifth points are undetermined in the projection; while of (c) there are, as we shall see below, six cases that are the part ingredients of degenerate quintics of the sheaves through the double lines, but which, however, are identical with six of the quartics of the nine sheaves of (a). So that, altogether, we have three sheaves and 123 isolated examples of unicursal plane quartics and nine sheaves of plane quartics, $d = 1$.

Each quartic of the nine sheaves meets eight simple lines once, while it fails to meet the ninth. This is as it should be, since the general plane section meets each line once, and all degenerate examples of such sections must preserve this property; and as we saw above that each cubic of the nine sheaves met one line and failed to meet the other eight, it becomes evident that the quartic associated to that cubic must meet those eight once and fail to meet that one. However, as we have already noted that in eight cases in each sheaf the cubic breaks up into a line and a conic, it is plain that in each sheaf there are eight quartics, each of which meets one line four times and one conic eight times.

In each sheaf there is one quartic that breaks up into a cubic and a line; we have already referred to this case when considering the associated cubics. It is that quartic of the sheaf whose projected cubic, passing through the eight fixed points of the sheaf, still passes through the ninth point, which point, as we know, is the center of the sheaf of lines that represent the associated cubics. This quartic, therefore, must break up into the cubic that is imaged by the cubic through the nine fundamental points and into the line imaged by the ninth point.

Except in the 72 cases noted above, however, each quartic of these nine sheaves, harmonious to its associated cubic, meets eight conics twice and the remainder once; while, counting the triple lines as isolated cubics, it meets 70 such cubics twice and the remainder once. Each such quartic, moreover, meets each of its companions of the same sheaf once, and of the other sheaves twice.

Moreover, each such quartic meets its associated cubic three times off of the multiple lines and, consequently, meets it nine times upon these lines. The cubic is unicursal, the quartic of deficiency one, and the section must possess three double and three triple points. Therefore, the intersection of these two curves must be such that the cubic has its node upon one triple line and the quartic its two nodes upon the other two triple lines; each curve passes through the node or nodes of the other and both still intersect each other upon the three double lines; the three remaining points of intersection are, consequently, points of contact of the plane of the section, and we thus obtain, at once, nine sheaves of tritangent planes. Moreover, there are 12 of such planes in each sheaf that are quartuply tangent; for Cayley, in the fourth volume of Crelle, p. 167, shows that the number of curves of a sheaf of $n^{\text{thic}}$ curves possessing $\alpha$ common double points, which still possess another double point, is

$$3(n-1)^2-7\alpha;$$

this, for $n=3$, $\alpha=0$, gives 12. We thus obtain (since this additional double point can be nothing else than the image of a point of contact of the plane containing the imaged quartic) 72 quarti-tangent planes.

Each of the 123 isolated quartics meets four lines once and fails to meet the remainder. Including the double lines, it meets 10 conics twice, 20 once and fails to meet the other six. It meets the cubics of five sheaves twice and those of the remaining four sheaves once. Of the associated quartics of these sheaves, it meets those of four sheaves thrice and those of five sheaves twice. It meets five of its own kind four times, 40 three times, 60 twice and 20 once, counting the special quartics in the planes through the triple lines. As was noted above and as will be shown below, in eleven cases these quartics break up into a triple line and a simple line.

Each of these quartics meets its associated cubic four times off of the multiple lines and, therefore, meets it eight times upon them. Since both curves are unicursal and since the section must have three double and three triple points, there are two classes of intersections, and the planes are so classified. (a). The quartic has its nodes upon the three triple lines; the cubic passes through these nodes, intersects the quartic still upon two double lines and has its node upon the third double line; there are, therefore, three varieties of this case. (b). The quartic has two of its nodes upon two of the triple lines, upon the third of which the node of the cubic lies; the two curves each pass through each other's node or nodes that lie upon the triple lines and intersect still upon two of the double lines, upon the third of which the third node of the quartic lies; there are nine varieties of this case. The four residue points of intersection of the two curves are, consequently, points of contact, and we thus obtain 123 new quarti-tangent planes.

Finally, we have the quartics of the three sheaves of planes through the triple lines, imaged as conics through four fundamental points. We have already seen that each triple line is met by five lines in such a manner that four of these fifteen lines are common to $A_1$ and $B_1$, one common to $A_1$ and $C_1$ and one common to $B_1$ and $C_1$, leaving the necessary remainder of nine. In such planes, therefore, the quartic breaks up into a cubic and a line, the cubic being imaged as a conic through five points; this fifth point, being a point of its intersection with the image of the axial triple line, is the image of the residue line. In four of these 15 planes, however (the four though, in fact, being but two counted twice), the plane becomes still further specialized, in that the cubic itself degenerates into a triple line; that is, the conic that images this cubic becomes identical with the conic that images the triple line in question. This is as it should be, since we know that $A_1$ and $B_1$ each meets $C_1$ and determine planes whose residues must be simple lines of the surface; these two planes, therefore, count once in each of the sheaves through $A_1$ and $B_1$ and twice in the sheaf through $C_1$. The remaining 11 planes are identical with the 11 cubo-quartic planes already referred to, in which the quartic breaks up into a triple line and a simple line.

Further, we have already seen that, including the double lines, each triple line is met twice by six conics; such planes, therefore, are determinate planes, in which the residue quartic breaks up into two conics. These cases, however, have already been treated in the discussion of the group of conics and notice taken that, since $A_1$ and $B_1$ each meet each double line once, while $C_1$ meets none, three conics group themselves with the double lines to form conic-pairs in the sheaf through $A_1$, three more similarly in the sheaf through $B_1$, while the remaining six group themselves in pairs in the sheaf through $C_1$, nine planes (three in each sheaf) being thus determined; in six of these planes, then (three through $A_1$ and three through $B_1$), the quartic breaks up into a double line and a conic and in the remaining three (those through $C_1$) it breaks up into two undegenerate conics.

The relations that these quartics bear to the simple lines, conics, plane cubics and other plane quartics of the surface, are the same as those already noted in the case of the 123 isolated quartics of the surface.

The quartics of the sheaves through $A_1$ and $B_1$ have a triple point that lies, in the one case, upon $B_1$ and, in the other case, upon $A_1$, while the axis meets $C_1$ and the three double lines. Since the section is thus complete and the quartic meets its axis four times, all such planes have triple contact along each of four sheets of the surface.

The quartics of the sheaf through $C_1$ are trinodal, having a node upon each of the double lines, while the axis meets $A_1$ and $B_1$; the section being thus complete and the quartic meeting its axis four times, these planes also have triple contact along each of four sheets of the surface. So that, altogether, the surface possesses three sheaves of planes of this kind.

We have still to examine the special planes of these sheaves.

The six planes, each of whose sections consists of a triple line, a double line and a conic, have a sextuple contact along one sheet of the surface, triple contact along two sheets and double contact along a fourth. This can best be seen by an example. Let the section be imaged by $A_1'$, $D_1'$ and 67. This last meets $A_1'$ twice, but only meets $D_1'$ once ulterior to the fundamental points; but since the conic imaged by 67 lies in the same plane with $D_1$, it must meet $D_1$ twice; one intersection must have been, therefore, absorbed in the projection of the multiple lines; but the only multiple line, besides $A_1$, that is met by $D_1$ is $B_1$; consequently, one point of intersection of the conic with $D_1$ must be at the point where $D_1$ meets $B_1$; in fact, when we consider the complete section, we see that this must be so. The axis $A_1$ meets $C_1$, $E_1$ and $F_1$, while $D_1$ and the conic meet $B_1$; this completes the section and leaves the intersection of $A_1$ and $D_1$, the two intersections of $A_1$ and the conic and the remaining intersection of $D_1$ and the conic to be reckoned as points of contact. We thus obtain the species of contact noted in the first sentence of this paragraph.

The three planes, whose sections each consists of a triple line and two undegenerate conics, can be, in the same manner, shown to have triple contact along four sheets of the surface and ordinary contact along a fifth sheet. $C_1$, the axis, meets both $A_1$ and $B_1$, while the two conics intersect upon the double lines; this leaves, for points of contact, the fourth point of intersection of the two conics and the four points in which the conic meets $C_1$. We see that three points of intersection of the two conics are absorbed in the projection of the double lines, as is also evident from the plane representation.

The two planes, whose sections each consists of two triple lines and a simple line, have contact still more highly singular. In both cases, $C_1$ is an ingredient and, therefore, meets the third triple line, while the second triple ingredient meets the three double lines and thus completes the section. The intersections of the three ingredients thus remain as points of planar contact. Each of these planes, therefore, has a 9-tuple contact along one sheet of the surface and triple contact along two other sheets.

There remains, to be examined, those planes, whose sections of the surface each consists of a triple line, a simple line and a nodal cubic, four of which occur in each of the sheaves through $A_1$ and $B_1$ and three in the sheaf through $C_1$.

The first eight planes have triple contact along four sheets of the surface and ordinary contact along a fifth sheet. An example will plainly reveal this. Let the section be imaged by 1 2 3 4 5, 4 6 7 8 9, 4. The axis $A_1$ meets

$C_1$, $D_1$, $E_1$, $F_1$, accounting for all the multiple points of the section with the exception of the triple point upon $B_1$; as $A_1$ does not meet $B_1$, the node of the cubic must lie upon $B_1$ at the point where the line of the section meets $B_1$ (this is also evident from the plane representation); there remains, consequently, for points of planar contact, the intersections of $A_1$ with the cubic and the simple line and the remaining intersection of these last two.

The three remaining planes can be, in the same manner, shown to have a triple contact along four sheets and ordinary contact along a fifth sheet. Of such singular planes we have, therefore, altogether fourteen examples.

We have now to consider the plane quintics, of which 21 isolated examples complete the sections through the 21 unabsorbed conics, while the remainder are the residues of the three sheaves of planes through the double lines.

The planes of the 21 quintics are tritangent planes and are of two kinds. The quintics, being of deficiency one, have five double points or one triple point and two double points. In the one species of plane, the quintic has three nodes upon the triple lines and the two nodes upon the two double lines; the conic passes through the nodes upon the triple lines and intersects the quintic still again upon the third double line. There are, consequently, three varieties of this species of plane. In the second species of plane, the quintic has a triple point upon one triple line and double points upon the other two triple lines; the conic passes through the two double points and still intersects the quintic upon each of the three double lines. There are, then, also three varieties of this species of plane.

Each one of these isolated quintics meets seven lines and fails to meet the other two; including the triple lines, it meets 21 of the isolated cubics once, 70 twice, and 35 three times, while it meets the associated quartics analogously; it meets each cubic of seven sheaves twice and of two sheaves thrice, while it meets the associated quartics analogously; it meets the quartics of the three sheaves in exactly the same manner as it meets the other unicursal quartics.

The quintics of the three sheaves have precisely the same relations to the simple lines and plane curves of the surface as the quintics just discussed. There are, however, to each sheaf, two planes in which the quintic breaks up into a line and a quartic. These six planes have already been remarked, in the discussion of the plane quartics, as planes in which the cubic broke up into a conic and a line. Such quartics are, consequently, common to the sheaves in question.

These special planes are planes having double contact along three sheets of the surface and ordinary contact along a fourth sheet, while the general planes of the sheaves each has double contact alone along three sheets of the surface. The contact takes place as follows: In the general plane, the axis meets both $A_1$ and $B_1$; through these points the quintic passes, has a triple point upon $C_1$ and two double points upon the two double lines; the remaining three points of intersection of the axis and the quintic are points of double contact. In the six special planes, the axis meets $A_1$ and $B_1$; the quartic passes through these points and has one double point upon $C_1$ and another double point upon one of the two remaining double lines; the residual line passes through the node upon $C_1$ and meets the quartic again upon the second remaining double line; the two residue intersections of the axis and the quartic, the intersection of the axis and the line and the remaining intersection of the quartic and the line are thus left as points of double and simple contact.

Of plane sextics, there are nine sheaves of deficiency, $d = 2$, being the residues of the sheaves of planes through the simple lines of the surface; such sextics are imaged as nodal quartics through the nine fundamental points with the node at the image of the axial line. These planes are of two kinds. In five of the nine sheaves, the section is of such sort that the sextic has a triple point upon one triple line and double points upon the other multiple lines; the line of the section passes through the two double points of the sextic that lie upon the triple lines. In the remaining four sheaves, the section is of such sort that the sextic has a triple point upon $A_1$ and $B_1$ and a double point upon $C_1$ and one double line, while the line of the section passes through the node at $C_1$ and still intersects the sextic upon the two remaining double lines. In both of these kind of planes, therefore, the line still meets the sextic twice, the points of intersection being the two points of contact of the plane (imaged by the node of the imaging quartic). This quartic may, however, have another double point, in which case the sextic of the surface has another node and the plane becomes tritangent; in each sheaf, therefore, there are 20 tritangent planes, giving, altogether, 180 new planes of this sort.

Each sextic of these sheaves meets the eight remaining lines once; counting the double lines, it meets eight conics once and the remainder twice, while it meets the associated quintics analogously; it meets the cubics of eight sheaves thrice and those of the other sheaf twice and the associated quartics analogously; it meets 70 of the isolated cubics twice and the remainder three times, and it meets the associated quartics analogously. Each sextic of the sheaf meets each of its kind of its own sheaf six times and, of the other sheaves, five times.

Finally, we have a quintuply-infinite sheaf of plane septics, of which all the above-enumerated curves form degenerate examples. These curves are the curves of the general plane section, and their properties have already been remarked.

Of twisted curves we have, upon the surface, of order not greater than six, the following: Nine sheaves of twisted cubics, imaged as lines through one fundamental point, and 126 isolated twisted cubics imaged as conics through five fundamental points; one net, 126 sheaves and 162 isolated examples of twisted quartics of the second kind, imaged, respectively, by the arbitrary lines of the plane, the conics through four fundamental points and the nodal cubics through seven; nine sheaves of twisted quartics of the first kind, imaged by non-singular cubics through eight fundamental points; 84 nets, 504 sheaves and 513 isolated examples of twisted quintics, $d = 0$, imaged, respectively, by conics through three points, nodal cubics through six points, and by 504 trinodal quartics through eight points and nine triple-pointic quartics through nine points; 36 nets and 36 sheaves of twisted quintics, $d = 1$, imaged, respectively, by non-singular quartics through seven and by binodal quartics through nine fundamental points; 36 webs, 630 nets, 1332 sheaves and 756 isolated examples of twisted sextics, $d = 0$, imaged, respectively, by conics through two points, nodal cubics through five points, 1260 trinodal quartics through seven and 72 triple-pointic quartics through eight points and 504 triple-pointic and trinodal quintics through nine and 252 6-nodal quintics through eight points; 84 webs, 288 nets and 126 sheaves of twisted sextics, $d = 1$, imaged, respectively, by non-singular cubics through six, binodal quartics through eight and 5-nodal quintics through eight points; and nine webs of twisted sextics, $d = 2$, imaged by nodal quartics through nine fundamental points. There are no twisted quintics of deficiency greater than one and no twisted sextics of deficiency greater than two upon the surface.

The twisted cubics meet the simple lines, the multiple lines and the various plane curves of the surface in the same manner as the plane unicursal cubics do; e. g. four sheaves of the twisted cubics meet $A_1$ and $B_1$ once, $C_1$ twice and the double lines each once; those of one sheaf meet $A_1$ twice and the other multiple lines each once; those of the sixth sheaf meet $B_1$ twice and each of the other multiple lines once, while those of the remaining three sheaves meet $A_1$ and $B_1$ twice, $C_1$ once and either $D_1$, $E_1$ or $F_1$ once, respectively.

Each quartic of the net fails to meet any line of the surface and, therefore, meets each associated sextic four times; it meets each conic once and the associated quintic and each quintic of the three sheaves, analogously; it meets each cubic of the nine sheaves once and the associated quartic analogously; it meets each isolated cubic twice and the associated quartic and each quartic of the three sheaves analogously; it meets each triple line twice and each double line once; and it meets the twisted cubics in the same way as it does their plane correspondents. Any two of these quartics meet once.

The other twisted quartics of the surface meet all other curves of the surface, just as the plane quartics, of the same species, do.

Each quintic of the 84 nets meets three lines once and fails to meet the remainder and, therefore, meets the plane sextics of three sheaves four times and those of six sheaves, five times; counting the double lines, it fails to meet three conics, meets 18 once and the remainder twice, while it meets the associated quintics and the quintics of the three sheaves analogously; it meets each of the cubics of three sheaves once and those of the remaining six sheaves twice and, consequently, meets the associated quartics analogously; counting the triple lines, it meets 15 of the isolated cubics once, 60 twice, 45 thrice and the remainder four times, while it meets the associated quartics and the quartics of the three sheaves analogously. Each triple line is met once by each of the quintics of 10 nets, twice by each of those of 40 nets, thrice by each of those of 30 nets and four times by each of those of the remaining four nets. It meets each twisted quartic of the net twice, and the other twisted curves, previously mentioned, are met by it in the same manner as their correspondents in the plane sections are. Each quintic of these 84 nets meets each of its own kind as follows: it meets each of those of the same net once, each of those of 18 nets twice, each of those of 45 nets thrice and each of those of the remaining 20 nets, four times.

Each quintic of the 504 sheaves meets one line twice, five lines once and fails to meet the remaining three; it meets the associated sextics analogously; counting the double lines, it meets three conics thrice, 15 twice, 13 once and fails to meet the remaining five; it meets the associated quintics and the quintics of the three sheaves analogously; it meets each of the cubics of one sheaf once, each of those of five sheaves twice, and each of those of the remaining three sheaves thrice; it meets the associated quartics analogously; counting the triple lines, it meets ten of the isolated cubics four times, 35 thrice, 45 twice, 31 once and fails to meet the remaining five; it meets the associated quartics and the quartics of the three sheaves analogously; it meets each twisted quartic of the net three times; it meets each twisted quintic of ten nets twice, each of those of 25 nets thrice, each of those of 33 nets four times, each of those of 15 nets five times, and each of those of the remaining net six times; it meets all other previously mentioned twisted curves in the same manner in which it meets their correspondents in the plane sections. Each triple line is met not at all by each of the quintics of 20 sheaves, is met once by each of those of 124 sheaves, is met twice by each of those of 180 sheaves, is met thrice by each of those of 140 sheaves, and is met four times by each of those of 40 sheaves. Each double line is met not at all by each of the quintics of 70 sheaves, is met once by each of those of 182 sheaves, is met twice by each of those of 210 sheaves, and is met thrice by each of those of 42 sheaves. Each quintic of these 504 sheaves meets each of its own kind, of its own sheaf, twice and each of those of the other sheaves from three to seven times, according to the number of coordinated lines they have in common.

The remaining loci of the surface and their configuration with the other curves of the surface can be easily calculated along the lines already followed.

The general plane section of *the surface*, $S^{(7)}_{1^3,\,5^2}$, consists of a septic curve possessing one triple and five double points and is imaged, in the simplest plane representation, by a septic curve which passes through 19 fixed points of which one is triple and five are double for the curve; these are the fundamental points of the projection. This surface possesses, then, only the 13 lines imaged by the 13 simple fundamental points and which, therefore, must be double chords of the double quintic that still meet the triple line; this is also apparent from the plane representation. These lines do not meet each other.

Since every plane through the triple line cuts out a quartic curve and since there is only one sheaf of plane quartics, imaged by the lines of the plane that center at the triple fundamental point, the image of the triple line must be the sextic curve, $1^2 . 2^2 \ldots . 6^2 . 7 . 8 \ldots . 19$, and, consequently, the image of the double quintic must be the 12thic, $1^6 . 2^3 . 3^3 \ldots . 6^3 . 7^2 . 8^2 \ldots . 19^2$, where 1 represents the triple fundamental point, 2 . . . . 6 represent the double fundamental points and 7 . . . . 19 represent the simple fundamental points.

The five double fundamental points and the five lines that join the triple to the double fundamental points image the ten conics possessed by the surface; these conics, consequently, occur in pairs, being, in fact, degenerate cases of the plane quartics of the sheaf. These conics do not meet the lines of the surface, but are each met by the double quintic thrice and each, of course, meets the triple line twice.

Since the conics occur in pairs as degenerate cases of the quartics of the sheaf, and since the surface possesses no other plane quartics than these, the plane cubics possessed by the surface can only occur in the 13 planes determined by the triple line and a simple line, being the residual sections of the surface by such planes; such planes, therefore, are degenerate planes of the sheaf of planes through the triple line, in which the residual quartics break up into lines and cubics. These 13 plane cubics are imaged, then, by the lines of the plane that join the triple fundamental point to the simple fundamental points.

Since no two lines of the surface meet, and since all conics occur in pairs, the surface can possess no plane quintics.

Of plane sextics, however, there are 13 sheaves, being the residual sections of the surface by the planes having for axes the simple lines of the surface; such are, consequently, imaged by septic curves through all the fundamental points, having triple points at the triple point, double points at the double points, and a sixth double point at the image of the axial line. In each sheaf, one plane breaks up into the triple line and a cubic, being the planes already mentioned.

These curves, with the septics of the general plane section, complete the list of plane curves possessed by the surface. Of such septics, there is a single sheaf.

Each cubic of the 13 planes fails to meet any line except the one coordinated to itself, and fails to meet any other plane curve except the sextics, each of which it meets three times. It meets the double quintic four times.

No plain quartic meets a line nor a plane curve except the sextics, each of which it meets four times. It meets the double quintic six times.

Each plane sextic meets each of the residue 12 lines once, each conic twice, each cubic thrice and each quartic four times. Two sextics of the same sheaf meet four times, of different sheaves five times. Each sextic meets the double quintic eight times and the triple line four times.

Each sextic and its associated line meet twice off of the multiple curves and four times upon them; the sextic has a double point upon the triple line and three more upon the double quintic; the line passes through the node upon the triple line and meets the sextic still twice upon the double quintic. Of these 13 sheaves of bitangent planes, there will be, after the formula of Cayley, 312 that are tritangent.

The general planes of the sheaf of planes through the triple line have triple contact along four sheets of the surface since the triple line meets the double quintic twice and the quartic residue has its three nodes upon the double quintic; the four points of intersection of the triple line and the quartic are points of planar contact.

In the 13 special planes of this sheaf in which the quartic degenerates into a line and a cubic, the triple line meets the double quintic twice, as before, while the nodal cubic has its node upon the double quintic and still meets the associated line twice upon that curve; the intersections of the triple line with the simple line and with the cubic and the third intersection of these last two with each other are all points of planar contact; these 13 planes, thus, have triple contact along four sheets of the surface and ordinary contact along a fifth.

The five planes in which the quartic degenerates into a pair of conics have, similarly, triple contact along four sheets and ordinary contact along a fifth. There are, therefore, altogether 18 planes which possess this order of singular tangency.

Of the twisted curves upon the surface, the rational twisted cubic which is imaged by the triple fundamental point is a distinguished one. This cubic fails to meet any line of the surface and only meets one conic of each pair. It meets each plane cubic and each plane quartic each once and meets each plane sextic twice. It meets the triple line twice and the double quintic six times.

Besides this distinguished cubic, the surface still possesses 15 other twisted cubics, of which 10 are imaged by the lines that join any two of the double fundamental points and the remaining five by conics passing through the triple and any four of the five double fundamental points. These 15 cubics behave in the same manner as the one already discussed.

The surface possesses 65 twisted quartics of the second kind, imaged by lines joining the double to the simple fundamental points, and one sheaf of such quartics imaged by the lines of the plane that center at the triple fundamental point. It still possesses 144 isolated examples of such quartics, of which one is imaged by the conic determined by the five double fundamental points, 130 imaged by conics determined by the triple, three double and one simple fundamental point and 13 imaged by cubics through all the multiple and one simple fundamental point having their nodes at the triple point.

The surface does not possess any twisted quartic of the first kind.

Of twisted quintics, $d=0$, the surface possesses 16 sheaves and 1378 isolated examples. Five sheaves are imaged by the sheaves of lines of the plane that have, for centers, the five double fundamental points; 10 sheaves are imaged by the conics of the plane that pass through the triple and any three double fundamental points; and the remaining sheaf is imaged by the nodal cubics that, having a node at the triple point, still pass through the five double points. Of the 1378 isolated examples, 78 are imaged by the lines that join the simple fundamental points, two by two; 65 are imaged by the conics that pass through four double and one simple fundamental point; 780 are imaged by the conics that pass through the triple, two double and two simple fundamental points; 65 are imaged by the cubics that, having a node at one double point, still pass through the remaining multiple points and one simple point; and 390 are imaged by the cubics that, having a node at the triple point, still pass through four double and two simple fundamental points.

Of twisted quintics, $d=1$, the surface possesses 286 isolated examples which are imaged by non-singular cubics determined by all the multiple and three simple fundamental points.

The surface possesses no twisted quintic of deficiency higher than one.

Of the 209 twisted quartics possessed by the surface, all but one each meet one line of the surface. This one, being the distinguished quartic imaged by the conic determined by the five double fundamental points, fails to meet any line of the surface, but meets each conic once, each plane cubic and each plane quartic twice, each plane sextic four times, fails to meet any twisted cubic and meets each other twisted quartic once. Of the remaining 208 twisted quartics, each of 65 meets four conics twice, two conics once and fails to meet the remainder and fails to meet any plane cubic; each of the residue 143 meets five conics once, fails to meet the other five, fails to meet one plane cubic and meets the other 12 each once. Each of these 208 quartics meets each quartic of the sheaf once, each sextic of 12 sheaves four times and each sextic of one sheaf three times; it meets 60 of its kind twice, 130 once and 18 twice; while each of 65 meets one twisted cubic twice, 10 once and fails to meet five, at the same time that each of 143 meets two twice, eight once and fails to meet six. Each of these quartics meets the triple line twice and the double quintic seven times, except that the distinguished quartic meets this last curve nine times.

No quintic of the sheaves meets a line of the surface. Each one meets one conic of each pair once but fails to meet its companion. Each meets each plane cubic once, each plane quartic once and each plane sextic five times. Each quintic of five sheaves meets one twisted cubic twice, 10 once and fails to meet five; each quintic of 10 sheaves meets one twisted cubic twice, eight once and seven not at all; while each quintic of the sixteenth sheaf meets every twisted cubic once. Each quintic of the five sheaves fails to meet 13 of the 65 twisted quartics but meets the remaining 52 once; it meets the distinguished quartic once, 79 of the 130 once and 52 twice, and it meets the 13 twice; each quintic of the 10 sheaves meets 39 of the 65 once and 26 twice; it meets the one once, it meets 39 of the 130 twice, 78 once and fails to meet 13, and it meets the 13 twice; each quintic of the sixteenth sheaf meets each of the 65 twisted quartics twice, meets the one once, each of the 130 once and fails to meet the 13.

Of the 1378 isolated twisted quintics, $d=0$, 1248 each meet two lines, each once, and 130 each meet one line once. Each of these same 1248 meets one conic from each pair, meets 11 cubics once, failing to meet the others, meets each quartic of the sheaf once, meets each sextic of 11 sheaves four times and each sextic of two sheaves three times; on the other hand, each of the above-mentioned 130 meets each conic of four pairs but fails to meet one conic of the fifth pair, meeting its companion twice; it meets 12 plane cubics twice and the thirteenth once; it meets each quartic of the sheaf twice; it meets each sextic of 12 sheaves five times and each sextic of the thirteenth sheaf four times. Each of the 1248 meets five twisted cubics twice, 10 once and fails to meet the sixteenth; it meets 11 twisted quartics thrice, 113 twice, 75 once and fails to meet 10; it meets each twisted quintic of one sheaf thrice, of 10 sheaves twice and of five sheaves once; on the other hand, each of the 130 meets eight twisted cubics once and fails to meet the other eight; it meets 96 twisted quartics twice, 104 once and fails to meet nine; it meets each quintic of eight sheaves twice and meets each quintic of the other eight sheaves once. Each quintic of these 1378 twisted quintics meets each other one once, twice or thrice according to their mutual relations to the lines and conics of the surface.

Each twisted quintic, $d = 1$, meets three lines of the surface, meets every conic, meets 10 plane cubics twice and three once, meets each quartic of the sheaf twice, meets the sextics of three sheaves four times and of 10 sheaves five times; it meets each of the twisted cubics once, 112 twisted quartics twice and 97 once, each quintic of the 16 sheaves twice, 720 of the 1378 isolated twisted quintics thrice, 580 twice and 78 once. Each of these 286 twisted quintics meets 30 of its kind once, 135 twice and 120 thrice.

The remaining loci of the surface and the relations that exist between themselves and between them and the other surface curves can be easily discussed along the lines already followed.

In *the surface*, $S^{(7)}_{3^2 . 4^2}$, the general sections consist of septic curves possessing two triple and four double points, imaged, in the simplest plane representation, by sextic curves, having in common five double and nine simple fundamental points.

These nine simple points image the only lines possessed by the surface; these lines are all non-planar, one being a chord of the triple conic and the other eight being chords of the double quartic that still meet the triple conic.

There are 16 simple conics upon the surface; five of these are imaged by the double fundamental points, 10 by the lines that join these points, two by two, and the sixteenth by the conic determined by these points. The first 15, consequently, form triplets or quadruplets in such a manner that while neither the first five meet each other nor the last 10 meet each other, these last are so associated to the first five that each of these five meets four of the 10 while each of the 10 meets two of the five; these 15 conics can be therefore grouped into 10 triplets or five quadruplets. The sixteenth conic meets each of the first five but fails to meet any of the last 10. Moreover each one of the 10-let joins itself to each one of two out of the 5-let to form a pair that is distinguished as being a degenerate example from the sheaves of trinodal twisted quartics that are imaged by the lines of the representative plane that center at the double fundamental points, four of such examples occurring in each sheaf. None of these simple conics meets a line of the surface.

The triple conic is imaged by the sextic $1^2 \ldots\ldots 5^2 . 6^2 . 7 \ldots\ldots 14$, where 6 is the image of the chord of the triple conic and the double quartic is imaged by the hyperelliptic $9^{thic}$, $1^3 \ldots . 5^3 . 7^2 \ldots . 14^3$. So that, as stated above, the remaining eight lines are the double chords of $C_4$ that still meet $C_2$.

Each of the simple conics meets the triple conic twice and the double quartic thrice.

There are no plane cubics or quartics upon the surface.

Since there are no line pairs upon the surface, the only plane quintics that the surface can possess are the 16 that are the residue sections of the planes through the 16 conics. Five of these are imaged by sextic curves through the 14 fundamental points having a triple point at the image of its associated conic and double points at the other four double fundamental points; 10 are imaged by quintic curves through all the fundamental points with double points at three of the double points; and the sixteenth quintic is imaged by the non-singular quartic that is determined by the 14 fundamental points. Each of these quintics is met by each line once.

The nine sheaves of plane sextics, whose planes have for axes the lines of the surface, are imaged by 6-nodal sextics through all the fundamental points with nodes at the five double and one simple fundamental point; in the sheaf through the line whose image is 6, one sextic degenerates into the triple conic of the surface. Each sextic meets each of eight lines once and each conic twice.

The configuration of the plane quintics among themselves is analogous to that of the conics among themselves. Two sextics of the same sheaf meet four times; of different sheaves, five times.

Of the plane septics of the general plane section, there is one web.

The nine sheaves of planes through the lines of the surface are, ordinarily, bitangent. The line meets the curve twice off of the multiple curves and four times upon them; the sextic is of deficiency four, while, in eight sheaves, the line is a double chord of the double quartic still meeting the triple conic. In such sheaves, therefore, the sextic has a triple and a double point upon the triple conic and two double points upon the double quartic, while it meets the line upon the double conic at the double point and twice again upon the double quartic; the residue intersections of the line and the sextic are points of contact; in each sheaf there are 33 planes that are tritangent. In the ninth sheaf the sextic is 6-nodal; two nodes, through which the line passes, are upon the triple conic and the other four upon the double quartic; in this sheaf there are also 33 tritangent planes, making altogether, so far, 297 tritangent planes.

The planes of the conics and the quintics, which curves meet seven times upon the multiple curves and three times off of them, are tritangent; the quintics, being of deficiency three, have two double points upon the triple conic and one upon the double quartic; the associated conic passes through the nodes upon the triple conic and still meets the quintic thrice upon the quartic; the three remaining points of intersection of the conic and the quintic are points of planar contact. This makes, altogether, 313 tritangent planes.

The plane of the triple conic has triple contact along two sheets of the surface.

There are 90 isolated twisted cubics upon the surface, of which 45 are imaged by the lines joining the double to the simple fundamental points and 45 by the conics determined by four double and one simple fundamental point. This completes the list of cubics possessed by the surface.

There are 10 sheaves and 576 isolated examples of twisted quartics of the second kind. Five sheaves are imaged by the lines that center at the double fundamental points and five by the conics that pass through four double fundamental points; four times in each sheaf, a quartic breaks up into a pair of non-planar conics and nine times into a non-planar cubic and a line. Thirty-six of the isolated quartics are imaged by the lines that join the simple fundamental points two by two; they are special examples out of sheaves of quintics; 360 are imaged by conics passing through three double and two simple fundamental points, and are also special examples out of sheaves of quintics; and 180 are imaged by nodal cubics through the five double and two simple fundamental points, node at a double point.

There are 126 isolated twisted quartics of the first kind, imaged by non-singular cubics through all the double and four simple fundamental points.

Of twisted quintics, $d = 0$, there are 144 sheaves and 3452 isolated examples. Nine sheaves are imaged by the lines that center at the simple fundamental points, 90 sheaves by the conics through three double and one simple fundamental point and 45 sheaves by nodal cubics passing through all the double and one simple fundamental point. Of the isolated quintics, 840 are imaged by conics through two double and three simple fundamental points; 1680 by nodal cubics through four double and three simple fundamental points, with node at a double point; 72 by nodal cubics through all the double and two simple fundamental points, having the node at the simple point; and 840 by trinodal quartics through all the double and three simple fundamental points, with the three nodes at three double points.

Of twisted quintics, $d = 1$, there are 84 sheaves, imaged by non-singular cubics through all the double and three simple fundamental points and 1890 isolated examples, of which 630 are imaged by non-singular cubics through four double and five simple fundamental points and 1260 are imaged by binodal quartics through the five double and five simple fundamental points.

Of twisted quintics, $d = 2$, there are 180 isolated examples, imaged by uninodal quartics through all the double and seven simple fundamental points, node at a double point.

A distinguished twisted curve of the surface is the $9^{\text{thic}}$ imaged by the cubic determined by the nine simple fundamental points.

Each twisted cubic meets some one line of the surface once. Each of the first 45 meets 12 conics once and fails to meet the other four, while each of the second 45 meets each of eight conics once and fails to meet the other eight. Each of the first 45 meets each of 12 plane quintics twice and each of four thrice, while each of the second 45 meets each of eight plane quintics twice and each of the other eight thrice. Each of these 90 twisted cubics meets the sextics of one sheaf twice and those of the remaining eight sheaves thrice; the sextics that are met but twice have for their axis the line that is met by the cubic considered. Each twisted cubic of each set of 45 meets 32 of its own set once and fails to meet the other 12, while it meets, of the other set, eight twice, 33 once, and four not at all. Each twisted cubic meets the double quartic four times, while 80 meet the triple conic thrice, the other 10 meeting it twice.

Each quartic of the 10 sheaves of twisted quartics fails to meet any line of the surface; it meets eight conics once and fails to meet the other eight; it meets the plane quintics and the plane sextics in a manner analogous to the way in which it meets, respectively, the conics and the lines; it meets the triple conic four times and the double quartic six times; it fails to meet any of its own sheaf but meets each quartic of eight sheaves once and each quartic of the remaining sheaf twice.

Each quartic of the 576 isolated quartics, $d = 0$, meets some two lines of the surface; it meets one conic twice, 10 once and fails to meet five; it meets the plane quintics and the plane sextics of the surface in a manner analogous to the way in which it meets, respectively, the conics and the lines; the triple conic is met by 128 of these quartics three times, and is met four times by the remainder, while each of the 576 quartics meets the double quartic five times; each quartic of these isolated quartics meets 10 twisted cubics twice, 70 once and fails to meet the remaining 10; it meets each twisted quartic of five sheaves twice and each one of the other five sheaves once; and, finally, it meets, of its own kind, 105 thrice, 280 twice, 166 once and fails to meet 24.

Each of the 126 twisted quartics, $d=1$, meets four lines of the surface; it meets each conic once; it meets the plane quintics and the plane sextics in a manner analogous to the way in which it meets, respectively, the conics and the lines; it meets 50 of the 90 twisted cubics twice and 40 once; it meets each quartic of the 10 sheaves twice; it meets, of the isolated twisted quartics, $d=0$, 160 thrice, 320 twice and 96 once. Each of these 126 twisted quartics, $d=1$, meets, of its own kind, 5 four times, 20 thrice, 60 twice and 40 once. It meets the double quartic four times, while the triple conic is met four times by 70 and three times by the remaining 56.

Each quintic of the 144 sheaves of twisted quintics, $d=0$, meets a single line of the surface while, of the 16 conics, it meets one twice, 10 once and fails to meet five; it meets the plane quintics and the plane sextics of the surface in a manner analogous to the way in which it meets, respectively, the conics and the lines of the surface; each quintic of the nine sheaves meets 80 twisted cubics once and fails to meet 10; each quintic of the 90 sheaves meets 32 twisted cubics twice, 52 once and fails to meet six, and each quintic of the 45 sheaves meets 64 twisted cubics twice, 24 once and fails to meet two; each quintic of these 144 sheaves meets each twisted quartic of five of the 10 sheaves twice and each one of the other five sheaves once; each of these quintics meets 140 of the isolated twisted quartics, $d=0$, thrice, 320 twice, 108 once, while it fails to meet eight; it meets, of the 126 isolated twisted quartics, $d=1$, 70 thrice and 56 twice; it meets the double quartic seven times, while the triple conic is met by the quintics of three sheaves four times and by each of those of the remaining sheaves five times; and, finally, each quintic of each sheaf fails to meet any quintic of its own sheaf, but of the quintics of the remaining 143 sheaves, it meets each one of 40 sheaves thrice, each one of 85 sheaves twice and each one of 18 sheaves once.

The relations borne by the remaining enumerated twisted curves, as well as the number, configurations and relations of those not enumerated, can be easily calculated and expressed by a continuation of the processes already employed.

In conclusion, it is only necessary to state that the ordinary singularities of these surfaces, such as the class and rank, the number of inflexional tangents, the class of the developable, the number of double tangents, the class of the spinode-torse and of the node-couple-torse, etc., are readily calculated from the general formulæ.

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# *On Sylvester's Proof of the Reality of the Roots of Lagrange's Determinantal Equation.*

BY THOMAS MUIR, LL. D.

(1). Of the various proofs that the roots of the equation

$$\begin{vmatrix} a_1 - x & a_2 & a_3 & \cdots \\ b_1 & b_2 - x & b_3 & \cdots \\ c_1 & c_2 & c_3 - x & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$$

are all real when the determinant is axisymmetric none is more elegant than Sylvester's, which appeared originally in the Philosophical Magazine, IV, pp. 138–142 (1852). In this account it is desirable to examine whether it be not readily applicable to prove the extension of the theorem which has been recently formulated,* viz. that

*The $n^{thic}$ equation*

$$\begin{vmatrix} 11 - x & 12 & 13 & \cdots \\ 21 & 22 - x & 23 & \cdots \\ 31 & 32 & 33 - x & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$$

*will have all its roots real if in the case of every pair $\mu$, $\nu$ of the $n-1$ indices* $2, 3, 4, \ldots, n$ *we have*

$$1\mu \,.\, \mu\nu \,.\, \nu 1 = \mu 1 \,.\, \nu\mu \,.\, 1\nu$$

*and* $$1\mu \,.\, \mu 1 = +.$$

(2). For shortness' sake, let us consider the equation of the 4th degree, viz.

$$\begin{vmatrix} a_1 - x & a_2 & a_3 & a_4 \\ b_1 & b_2 - x & b_3 & b_4 \\ c_1 & c_2 & c_3 - x & c_4 \\ d_1 & d_2 & d_3 & d_4 - x \end{vmatrix} = 0,$$

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* Philos. Mag.

in which case we have to establish the reality of all the roots when it is given that

$$\begin{aligned} a_2b_3c_1 &= a_3b_1c_2\,, & a_2b_1 &= +, \\ a_2b_4d_1 &= a_4b_1d_2, & a_3c_1 &= +, \\ a_3c_4d_1 &= a_4c_1d_3, & a_4d_1 &= +, \end{aligned}$$

it being noted however that these three conditional equations imply another, viz.

$$b_3c_4d_2 = b_4c_2d_3\,,$$

and that, along with the three conditions as to sign, they also imply three others of the latter character, viz.

$$\begin{aligned} b_3c_2 &= +, \\ b_4d_2 &= +, \\ c_4d_3 &= +, \end{aligned}$$

so that, in fact, the determinant is given axisymmetric as to sign.

Denoting, then, the above equation of the 4th degree by

$$f(x) = 0\,,$$

we have for the product of the two determinants $f(x)$ and $f(-x)$,

$$\left|\begin{matrix} \dfrac{a_1, a_2, a_3, a_4}{a_1, b_1, c_1, d_1} - x^2 & \dfrac{b_1, b_2, b_3, b_4}{a_1, b_1, c_1, d_1} & \dfrac{c_1, c_2, c_3, c_4}{a_1, b_1, c_1, d_1} & \dfrac{d_1, d_2, d_3, d_4}{a_1, b_1, c_1, d_1} \\ \dfrac{a_1, a_2, a_3, a_4}{a_2, b_2, c_2, d_2} & \dfrac{b_1, b_2, b_3, b_4}{a_2, b_2, c_2, d_2} - x^2 & \dfrac{c_1, c_2, c_3, c_4}{a_2, b_2, c_2, d_2} & \dfrac{d_1, d_2, d_3, d_4}{a_2, b_2, c_2, d_2} \\ \dfrac{a_1, a_2, a_3, a_4}{a_3, b_3, c_3, d_3} & \dfrac{b_1, b_2, b_3, b_4}{a_3, b_3, c_3, d_3} & \dfrac{c_1, c_2, c_3, c_4}{a_3, b_3, c_3, d_3} - x^2 & \dfrac{d_1, d_2, d_3, d_4}{a_3, b_3, c_3, d_3} \\ \dfrac{a_1, a_2, a_3, a_4}{a_4, b_4, c_4, d_4} & \dfrac{b_1, b_2, b_3, b_4}{a_4, b_4, c_4, d_4} & \dfrac{c_1, c_2, c_3, c_4}{a_4, b_4, c_4, d_4} & \dfrac{d_1, d_2, d_3, d_4}{a_4, b_4, c_4, d_4} - x^2 \end{matrix}\right|,$$

where $\dfrac{a,\ b,\ c,\ d}{m,\ n,\ p,\ q}$ is used for shortness' sake to denote the bipartite

$$(a,\ b,\ c,\ d\,\big)\!\big(\,m,\ n,\ p,\ q),$$

i. e. $am + bn + cp + dq$. Expanding this determinant according to descending powers of $x^2$, we obtain

$$
\begin{aligned}
&x^8 \\
&- x^6 \left\{ \frac{a_1 a_2 a_3 a_4}{a_1 b_1 c_1 d_1} + \frac{b_1 b_2 b_3 b_4}{a_2 b_2 c_2 d_2} + \frac{c_1 c_2 c_3 c_4}{a_3 b_3 c_3 d_3} + \frac{d_1 d_2 d_3 d_4}{a_4 b_4 c_4 d_4} \right\} \\
&+ x^4 \left\{
\begin{aligned}
& \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \\
+ & \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \\
+ & \begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} + \begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}
\end{aligned}
\right\} \\
&- x^2 \left\{
\begin{aligned}
& \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \\
& \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}
\end{aligned}
\right\} \\
&+ \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} ;
\end{aligned}
$$

and all that remains is to show that the coefficients of this expansion are all positive.

Now the coefficient of $-x^6$ is

$$
\begin{aligned}
& a_1^2 + a_2 b_1 + a_3 c_1 + a_4 d_1 \\
+ & b_1 a_2 + b_2^2 + b_3 c_2 + b_4 d_2 \\
+ & c_1 a_3 + c_2 b_3 + c_3^2 + c_4 d_3 \\
+ & d_1 a_4 + d_2 b_4 + d_3 c_4 + d_4^2 ,
\end{aligned}
$$

and every individual term of it is seen to be positive because of the sign-conditions.

The coefficient of $x^4$ is the axisymmetric square array of 36 terms:

$$
\begin{aligned}
& |a_1 b_2|^2 + |a_1 b_3||a_1 c_2| + |a_1 b_4||a_1 d_2| + |a_2 b_3||b_1 c_2| + |a_2 b_4||b_1 d_2| + |a_3 b_4||c_1 d_2| \\
+ & |a_1 c_2||a_1 b_3| + |a_1 c_3|^2 + |a_1 c_4||a_1 d_3| + |a_2 c_3||b_1 c_3| + |a_2 c_4||b_1 d_3| + |a_3 c_4||c_1 d_3| \\
+ & |a_1 d_2||a_1 b_4| + |a_1 d_3||a_1 c_4| + |a_1 d_4|^2 + |a_2 d_3||b_1 c_4| + |a_2 d_4||b_1 d_4| + |a_3 d_4||c_1 d_4| \\
+ & |b_1 c_2||a_2 b_3| + |b_1 c_3||a_2 c_3| + |b_1 c_4||a_2 d_3| + |b_2 c_3|^2 + |b_2 c_4||b_2 d_3| + |b_3 c_4||c_2 d_3| \\
+ & |b_1 d_2||a_2 b_4| + |b_1 d_3||a_2 c_4| + |b_1 d_4||a_2 d_4| + |b_2 d_3||b_2 c_4| + |b_2 d_4|^2 + |b_3 d_4||c_2 d_4| \\
+ & |c_1 d_2||a_3 b_4| + |c_1 d_3||a_3 c_4| + |c_1 d_4||a_3 d_4| + |c_2 d_3||b_3 c_4| + |c_2 d_4||b_3 d_4| + |c_3 d_4|^2
\end{aligned}
$$

where six of the terms are clearly positive and fifteen require examination. Taking the first of the latter, viz. $|a_1 b_3| . |a_1 c_2|$ and multiplying and dividing by $b_3$, we see that it

$$= |a_1 b_3| . (a_1 c_2 b_3 - a_2 c_1 b_3) \div b_3,$$

which, on account of the conditional equation $a_2 b_3 c_1 = a_3 b_1 c_2$,

$$= |a_1 b_3| . (a_1 b_3 - a_3 b_1) c_2 \div b_3,$$

$$= |a_1 b_3|^2 . \frac{c_2}{b_3},$$

and therefore is positive because $b_3$ and $c_2$ are given alike in sign. The same mode of proof suffices for all the others except the three in the second diagonal of the square, in which cases the introduced multiplier is the product of *two* elements. For example,

$$\begin{aligned} |a_3 b_4| . |c_1 d_2| &= |a_3 b_4| . (a_3 b_4 c_1 d_2 - a_3 b_4 c_2 d_1) \div a_3 b_4, \\ &= |a_3 b_4| . (a_3 b_4 c_1 d_2 - a_4 b_3 c_1 d_2) \div a_3 b_4, \\ &= |a_3 b_4|^2 . \frac{c_1 d_2}{a_3 b_4} \\ &= + \end{aligned}$$

because $a_3$ and $c_1$ are alike in sign, and also $b_4$ and $d_2$.

The coefficient of $-x^2$ is the axisymmetric square array of sixteen terms,

$$\begin{array}{llll} |a_1 b_2 c_3|^2 & + |a_1 b_2 c_4| \, |a_1 b_2 d_3| & + |a_1 b_3 c_4| \, |a_1 c_2 d_3| & + |a_2 b_3 c_4| \, |b_1 c_2 d_3| \\ + |a_1 b_2 d_3| \, |a_1 b_2 c_4| & + |a_1 b_2 d_4|^2 & + |a_1 b_3 d_4| \, |a_1 c_2 d_4| & + |a_2 b_3 d_4| \, |b_1 c_2 d_4| \\ + |a_1 c_2 d_3| \, |a_1 b_3 c_4| & + |a_1 c_2 d_4| \, |a_1 b_3 d_4| & + |a_1 c_3 d_4|^2 & + |a_2 c_3 d_4| \, |b_1 c_3 d_4| \\ + |b_1 c_2 d_3| \, |a_2 b_3 c_4| & + |b_1 c_2 d_4| \, |a_2 b_3 d_4| & + |b_1 c_3 d_4| \, |a_2 c_3 d_4| & + |b_2 c_3 d_4|^2 \end{array}$$

where four of the terms are clearly positive and six require examination. Taking the first of the latter, viz. $|a_1 b_2 c_4| \, |a_1 b_2 d_3|$ and multiplying and dividing by $c_4$, we obtain

$$|a_1 b_2 c_4| . \{ |a_1 b_2| d_3 c_4 - |a_1 d_2| b_3 c_4 + |b_1 d_2| a_3 c_4 \} \div c_4$$

which

$$\begin{aligned} &= |a_1 b_2 c_4| . \left\{ |a_1 b_2| d_3 c_4 - |a_1 b_4| . \frac{d_2}{b_4} b_3 c_4 + |a_2 b_4| \frac{d_1}{a_4} . a_3 c_4 \right\} \div c_4, \\ &= |a_1 b_2 c_4| . \{ |a_1 b_2| d_3 c_4 - |a_1 b_4| c_2 d_3 + |a_2 b_4| c_1 d_3 \} \div c_4, \\ &= |a_1 b_2 c_4| . |a_1 b_2 c_4| . \frac{d_3}{c_4} \\ &= + \end{aligned}$$

because $c_4$ and $d_3$ are alike in sign. For the five other terms, exactly the same mode of proof suffices.

Lastly, the coefficient of $x^0$ is manifestly $|a_1 b_2 c_3 d_4|^2$.

The theorem is thus established.

(3). There are several results obtained in the foregoing which are well worth fuller notice.

In the first place, it should be observed that not only is the coefficient of each power of $x$ positive, but it is shown to be so by reason of the fact that every term of it is positive, exactly as in Sylvester's original case.

Secondly, it should be noted that in dealing with the coefficient of $x^4$ we proved that each minor of the 2[d] order bears a simple ratio to its conjugate minor, viz.

$$
\begin{array}{l}
|a_1 b_3| : |a_1 c_2| :: b_3 : c_2 \;, \\
|a_1 b_4| : |a\ d_2| :: b_4 : d_2 \;, \\
|a_2 b_3| : |b_1 c_2| :: a_3 : c_1 \;, \\
|a_2 b_4| : |b_1 d_2| :: a_4 : d_1 \;, \\
|a_3 b_4| : |c_1 d_2| :: a_3 b_4 : c_1 d_2 \,, \\
|a_1 c_4| : |a_1 d_3| :: c_4 : d_3 \;, \\
|a_2 c_3| : |b_1 c_3| :: a_2 : b_1 \;, \\
|a_2 c_4| : |b_1 d_3| :: a_2 c_4 : b_1 d_3 \,, \\
|a_3 c_4| : |c_1 d_3| :: a_4 : d_1 \;, \\
|a_2 d_3| : |b_1 c_4| :: a_2 d_3 : b_1 c_4 \,, \\
|a_2 d_4| : |b_1 d_4| :: a_2 : b_1 \;, \\
|a_3 d_4| : |c_1 d_4| :: a_3 : c_1 \;, \\
|b_2 c_4| : |b_2 d_3| :: c_4 : d_3 \;, \\
|b_3 c_4| : |c_2 d_3| :: b_4 : d_2 \;, \\
|b_3 d_4| : |c_2 d_4| :: b_3 : c_2 \;.
\end{array}
$$

Thirdly, in dealing with the coefficient of $x^3$ it was shown that each minor of the 3[d] order bears a similarly simple ratio to its conjugate minor, viz.

$$
\begin{array}{l}
|a_1 b_2 c_4| : |a_1 b_2 d_3| :: c_4 : d_3 \,, \\
|a_1 b_3 c_4| : |a_1 c_2 d_3| :: b_4 : d_2 \,, \\
|a_2 b_3 c_4| : |b_1 c_2 d_3| :: a_4 : d_1 \,, \\
|a_1 b_3 d_4| : |a_1 c_2 d_4| :: b_3 : c_2 \,, \\
|a_2 b_3 d_4| : |b_1 c_2 d_4| :: a_3 : c_1 \,, \\
|a_2 c_3 d_4| : |b_1 c_3 d_4| :: a_2 : b_1 \,.
\end{array}
$$

(4). A little examination suffices to make evident that only six ratios are involved in all these proportions, viz. the ratio of each non-diagonal element to its conjugate. All the proportions except three can thus be combined as follows:

$$\frac{a_2}{b_1} = \frac{|a_2\,c_3|}{|b_1\,c_3|} = \frac{|a_2\,d_4|}{|b_1\,d_4|} = \frac{|a_2\,c_3\,d_4|}{|b_1\,c_3\,d_4|},$$

$$\frac{a_3}{c_1} = \frac{|a_3\,b_2|}{|c_1\,b_2|} = \frac{|a_3\,d_4|}{|c_1\,d_4|} = \frac{|a_3\,b_2\,d_4|}{|c_1\,b_2\,d_4|},$$

$$\frac{a_4}{d_1} = \frac{|a_4\,b_2|}{|d_1\,b_2|} = \frac{|a_4\,c_3|}{|d_1\,c_3|} = \frac{|a_4\,b_2\,c_3|}{|d_1\,b_2\,c_3|},$$

$$\frac{b_3}{c_2} = \frac{|b_3\,a_1|}{|c_2\,a_1|} = \frac{|b_3\,d_4|}{|c_2\,d_4|} = \frac{|b_3\,a_1\,d_4|}{|c_2\,a_1\,d_4|},$$

$$\frac{b_4}{d_2} = \frac{|b_4\,a_1|}{|d_2\,a_1|} = \frac{|b_4\,c_3|}{|d_2\,c_3|} = \frac{|b_4\,a_1\,c_3|}{|d_2\,a_1\,c_3|},$$

$$\frac{c_4}{d_3} = \frac{|c_4\,a_1|}{|d_3\,a_1|} = \frac{|c_4\,b_2|}{|d_3\,b_2|} = \frac{|c_4\,a_1\,b_2|}{|d_3\,a_1\,b_2|}.$$

It therefore appears that we can formulate the following general theorem: *If the determinant of the* $4^{th}$ *order* $|a_1\,b_2\,c_3\,d_4|$ *fulfil the specified conditions, the ratio of any non-diagonal element pq to its conjugate qp*

$$= \frac{|pq\,.\,rr|}{|qp\,.\,rr|} = \frac{|pq\,.\,ss|}{|qp\,.\,ss|} = \frac{|pq\,.\,rr\,.\,ss|}{|qp\,.\,rr\,.\,ss|}.$$

In the case of the three excepted proportions the ratios which occur are compounded of these six, viz.

$$\frac{a_2\,c_4}{b_1\,d_3} = \frac{a_4\,c_2}{b_3\,d_1} = \frac{|a_2\,c_4|}{|b_1\,d_3|},$$

$$\frac{a_3\,b_4}{c_1\,d_2} = \frac{a_4\,b_3}{c_2\,d_1} = \frac{|a_3\,b_4|}{|c_1\,d_2|},$$

$$\frac{a_2\,d_3}{b_1\,c_4} = \frac{a_3\,d_2}{b_4\,c_1} = \frac{|a_2\,d_3|}{|b_1\,c_4|}.$$

(5). Another important property of the determinant $|a_1\,b_2\,c_3\,d_4|$ is that *every term is equal to its conjugate.*

To establish this we have to prove seven identities, for of the total twenty-four terms ten are known* to be self-conjugate. The first four

* Proc. Roy. Soc. Edinburgh, XVII, pp. 7–18.

$$a_2 b_3 c_1 d_4 = a_3 b_1 c_2 d_4,$$
$$a_2 b_4 c_3 d_1 = a_4 b_1 c_3 d_2,$$
$$a_3 b_2 c_4 d_1 = a_4 b_2 c_1 d_3,$$
$$a_1 b_3 c_4 d_2 = a_1 b_4 c_2 d_3,$$

are readily got from the conditional equations by multiplying by $d_4$, $c_3$, $b_2$, $a_1$ respectively. Each of the remaining three is got from a *pair* of the conditional equations by multiplication and division, thus—

from the 1st and 4th equations

$$a_2 b_3 c_1 . b_4 c_2 d_3 = a_3 b_1 c_2 . b_3 c_4 d_2,$$

and $\therefore$ $$a_2 b_4 c_1 d_3 = a_3 b_1 c_4 d_2;$$

from the 2d and 4th equations

$$a_2 b_4 d_1 . b_3 c_4 d_2 = a_4 b_1 d_2 . b_4 c_2 d_3$$

and $\therefore$ $$a_2 b_3 c_4 d_1 = a_4 b_1 c_2 d_3;$$

and from the 3d and 4th equations

$$a_3 c_4 d_1 . b_4 c_2 d_3 = a_4 c_1 d_3 . b_3 c_4 d_2$$

and $\therefore$ $$a_3 b_4 c_2 d_1 = a_4 b_3 c_1 d_2.$$

(6). Finally, it should be noted that the conditions here given for the reality of the roots of $f(x) = 0$ are exactly the conditions for the determinant $f(x)$ being transformable by multiplication of rows and columns into an axisymmetric determinant in which $x$ is involved in the same way as before the transformation.

# *Concerning the Twisted Biquadratic.*

By Dr. J. C. Kluyver, *Leyden.*

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Among the covariant figures of the twisted biquadratic $R$ there exists a determinate tetrahedral quartic surface $F$, whose relation to the curve will be the subject of the following paper.

Let $R$ be given as the intersection of two quadrics $S$ and $S'$, then, by a known theorem, the coordinates of a point on the curve are rational expressions of the elliptic functions, obtained by inverting the elliptic integral

$$u=\int_{\xi_0}^{\xi}\frac{d\xi}{\sqrt{\Delta'\xi^4+\theta'\xi^3+\phi\xi^2+\theta\xi+\Delta}}=\int_{\xi_0}^{\xi}\frac{d\xi}{\sqrt{G(\xi)}},$$

where $\Delta'$, $\theta'$, $\phi$, $\theta$, $\Delta$ denote in the usual manner the common invariants of $S$ and $S'$, and in this way to each point on the curve a determinate argument $u$ becomes affixed.

If we make the lower limit $\xi_0$ of the integral a root of $G(\xi)=0$, the doubly-periodic function $\xi(u)$ will be even, and therefore it will only be some linear transformation of Weierstrass's function $\wp u$. At the same time the derived function $\xi'(u)$ will be odd and $\xi'(u)=\pm\sqrt{G(\xi)}$ will have the four zeros $0, \omega_1, \omega_2, \omega_3$, where $\omega_1$, $\omega_2$, $\omega_3$ are three half-periods, satisfying the relation $\omega_1+\omega_2+\omega_3=0$.

Every plane now intersects $R$ in four points, the arguments of which have a sum equal to zero, and consequently on a quadric through $R$ every right line of the first system meets the curve in two points, whose arguments have the constant sum $v$, whereas for the right lines of the other system that sum is equal to $-v$. For the four cones, which pass through $R$, the two systems coincide, hence with these surfaces are associated the values $v=0, \omega_1, \omega_2, \omega_3$.

We can ask for the value of $v$, belonging to the quadric $S+\mu S'$. Evidently to a given value of $\mu$ there are corresponding two values of $v$, only differing in sign, conversely $v$ being given, only one value of $\mu$ can be found. From this we infer that $\mu$ is some linear function of $\xi(v)$, but as we have for the four cones $G(\mu)=0$, and as the roots of the same equation are precisely the values which $\xi(v)$ assumes for $v=0,\ \omega_1,\ \omega_2,\ \omega_3$, it is clear that the parameter $\mu$ of the quadric $S+\mu S'$ is simply equal to $\xi(v)$.

It is now possible to discriminate the quadrics of the system $S+\mu S'$ according to the different values of $v$, and in doing so, we are led to consider the surfaces corresponding to rational parts of the periods. More particularly, we may notice the six surfaces belonging to the quarter-periods, which, according to Ameseder,* were first studied by Voss.† We shall call, for shortness, these surfaces the Vossian quadrics of the curve $R$, and shall denote them by $H_1$, $H_{\rm I}$, $H_2$, $H_{\rm II}$, $H_3$, $H_{\rm III}$, the six associated values of $v$ being

$$\frac{\omega_1}{2},\ \frac{\omega_1}{2}+\omega_2;\ \frac{\omega_2}{2},\ \frac{\omega_2}{2}+\omega_3,\ \frac{\omega_3}{2},\ \frac{\omega_3}{2}+\omega_1.$$

From these we get only three distinct values of $2v$, therefore the Vossian quadrics may be arranged in three pairs of conjugate surfaces: $H_1$ and $H_{\rm I}$, $H_2$ and $H_{\rm II}$, $H_3$ and $H_{\rm III}$, each pair corresponding to one of the three half-periods. We next examine the values of the parameter $\mu$ for the Vossian quadrics. By the properties of the function $\wp u$ we know that if $\wp 0$, $\wp\omega_1$, $\wp\omega_2$, $\wp\omega_3$ are the roots of a binary quartic, the roots of its sextic covariant are the values of $\wp v$ for the quarter-periods. Now since $\xi(v)$ is only a linear transformation of $\wp v$, the same theorem holds good for $\xi(v)$, and so it is apparent that the required values of $\mu$ are the zeros of the sextic covariant $T(\mu)$, derived from the fundamental quartic $G(\mu)$.

Let us suppose that the two quadrics $S$ and $S'$ are a pair of conjugate Vossian quadrics, then 0 and $\infty$ form one of the three pairs of conjugate roots of $T(\mu)$, hence in the quartic $G(\mu)$ only even powers of $\mu$ occur, and we thus get the theorem: For each pair of conjugate Vossian quadrics the common invariants $\theta'$ and $\theta$ will vanish identically. Reciprocating this theorem, we can prove that as soon as the invariants $\theta'$ and $\theta$ are vanishing, the quadrics $S$ and $S'$ are a pair of conjugate Vossian quadrics with respect to their intersection.

It is easily shown, by means of the special values of $v$, that on any one of these Vossian quadrics we can place the sides of an infinity of skew quadrilate-

* "Ueber Configurationen auf der Raumcurve 4[ter] Ordnung, 1[ster] Species"; Wiener Sitzungsberichte, Bd. 87, p. 1194.

† "Die Liniengeometrie in ihrer Anwendung auf die Flächen 2[ten] Grades"; Math. Annalen, Bd. 10.

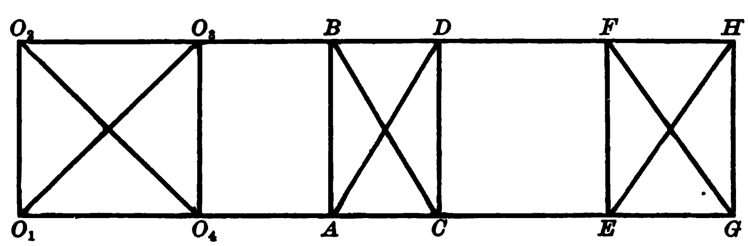
rals inscribed in the biquadratic $R$. Moreover we can find on each of these surfaces four chords $XY$ of the curve $R$, such that three times the argument of either extremity added to the argument of the other one is equal to zero. Otherwise stated, we can find on a Vossian quadric four chords $XY$, each of which is the intersection of the osculating planes in its extremities. So for example on $H_1$, corresponding to $v = \frac{\omega_1}{2}$, we have only to join the points $X$ and $Y$ with the affixes

$$X \ldots\ldots \frac{\omega_1}{4} + \omega_2, \quad -\frac{3\omega_1}{4} + \omega_2, \quad -\frac{\omega_1}{4} + \omega_2, \quad -\frac{3\omega_1}{4} + \omega_2,$$

$$Y \ldots\ldots \frac{\omega_1}{4} + \omega_3, \quad -\frac{3\omega_1}{4} + \omega_3, \quad -\frac{\omega_1}{4} + \omega_3, \quad -\frac{3\omega_1}{4} + \omega_3,$$

to obtain four chords $XY$ possessing this characteristic property.

In all we have 24 of these chords, which we shall call the *chords of curvature* of $R$. Their position with respect to $R$ and with respect to the self-conjugate tetrahedron $O_1 O_2 O_3 O_4$, associated with that curve, is very symmetrical, but as it is amply discussed in the cited paper of Ameseder and in other papers dealing with the theory of the biquadratic, it may suffice to mention here that the chords of curvature can be divided into three groups of 8, all chords of one group meeting the same pair of opposite edges of the tetrahedron $O_1 O_2 O_3 O_4$. In the accompanying schematical diagram one of these groups is represented; through four of the eight chords $AB$, $BC$, $CD$, $DA$, forming the sides of a skew




quadrilateral $ABCD$ whose vertices lie upon the opposite edges $O_2 O_3$ and $O_1 O_4$, the Vossian quadric $H_1$ can be described, similarly through the other four $EF$, $FG$, $GH$, $HE$ passes the conjugate quadric $H_{\text{I}}$. Further investigation proves that each of the three pairs of points $O_1$, $O_4$; $A$, $C$; $E$, $G$ on the edge $O_1 O_4$, likewise each of the three pairs $O_2$, $O_3$; $B$, $D$; $F$, $H$ on the edge $O_2 O_3$, may be regarded as the pair of foci of the involution defined by the two remaining pair of points. Thus, then, it is evident that the complete group of eight chords can be constructed without ambiguity as soon as one of them, $AB$ for instance, is placed at random on the edges $O_2 O_3$ and $O_1 O_4$ of the self-conjugate tetrahedron, which henceforth we shall use as the tetrahedron of reference. Let $AB$ be represented by the equations $x = aw$, $y = az$, where $a$ denotes $e^{\frac{\pi i}{4}}$, then we can write down the equations of all the chords of the group as follows:

$$AB\begin{cases} x = aw \\ y = az \end{cases}, \quad BC\begin{cases} x = -aw \\ y = az \end{cases}, \quad CD\begin{cases} x = -aw \\ y = -az \end{cases}, \quad DA\begin{cases} x = aw \\ y = -az \end{cases},$$

$$EF\begin{cases} x = a^3w \\ y = a^3z \end{cases}, \quad FG\begin{cases} x = -a^3w \\ y = a^3z \end{cases}, \quad GH\begin{cases} x = -a^3w \\ y = -a^3z \end{cases}, \quad HA\begin{cases} x = a^3w \\ y = -a^3z \end{cases}.$$

But though the eight chords are now defined, they fail to determine completely the equations of the Vossian quadrics $H_1$ and $H_{\text{I}}$ and of their intersection $R$. Introducing two arbitrary constants $\lambda$ and $\mu$, we only know that these equations can be thrown into the form

$$H_1 = \lambda(x^2 - iw^2) + (y^2 - iz^2) = 0,$$
$$H_{\text{I}} = \mu(x^2 + iw^2) + (y^2 + iz^2) = 0;$$

for each of these quadrics contains the four sides of one of the previously considered quadrilaterals, and, as an easy reckoning shows, the conditions $\theta' = 0$, $\theta = 0$, characteristic for a pair of conjugate Vossian quadrics, are fulfilled. Passing now from $H_1$ and $H_{\text{I}}$ to the remaining pairs of Vossian quadrics, we get without much difficulty for their equations

$$\begin{cases} H_2 = \sqrt{\mu}H_1 + i\sqrt{\lambda}H_{\text{I}} \\ \qquad = (\sqrt{\lambda} + i\sqrt{\mu})(x^2\sqrt{\lambda\mu} - z^2) + (\sqrt{\mu} + i\sqrt{\lambda})(y^2 - w^2\sqrt{\mu\lambda}) = 0, \\ H_{\text{II}} = \sqrt{\mu}H_1 - i\sqrt{\lambda}H_{\text{I}} \\ \qquad = (\sqrt{\lambda} - i\sqrt{\mu})(x^2\sqrt{\lambda\mu} + z^2) + (\sqrt{\mu} - i\sqrt{\lambda})(y^2 + w^2\sqrt{\mu\lambda}) = 0, \end{cases}$$

$$\begin{cases} H_3 = \sqrt{\mu}H_1 + \sqrt{\lambda}H_{\text{I}} \\ \qquad = (\sqrt{\lambda} + \sqrt{\mu})(x^2\sqrt{\lambda\mu} + y^2) + i(\sqrt{\lambda} - \sqrt{\mu})(z^2 - w^2\sqrt{\mu\lambda}) = 0, \\ H_{\text{III}} = -\sqrt{\mu}H_1 + \sqrt{\lambda}H_{\text{I}} \\ \qquad = (\sqrt{\lambda} - \sqrt{\mu})(-x^2\sqrt{\lambda\mu} + y^2) + i(\sqrt{\lambda} + \sqrt{\mu})(z^2 + w^2\sqrt{\mu\lambda}) = 0. \end{cases}$$

From these equations we infer that the 24 chords of curvature become perfectly defined, if we choose a determinate value for the as yet arbitrary constant $\sqrt{\mu\lambda}$, but obviously, by doing so, we do not succeed in determining the biquadratic $R$, because this curve is represented by the two equations of $H_1$ and $H_{\text{I}}$, wherein the constants $\lambda$ and $\mu$ occur separately. Hence we have arrived at the theorem: There is an infinity of biquadratic curves that have the same chords of curvature in common with a given curve $R$.

Let us take $\sqrt{\mu\lambda} = \alpha^2$, then we are at once enabled to derive from the equations of $H_1$ and $H_I$ the locus of these biquadratics. Any one of them is now represented by

$$\begin{cases} \lambda(x^2 - iw^2) + (y^2 - iz^2) = 0, \\ (x^2 + iw^2) - \lambda(y^2 + iz^2) = 0, \end{cases}$$

where $\lambda$ is a variable parameter, and hence, by eliminating $\lambda$, we find the required locus to be the tetrahedral quartic surface

$$F = x^4 + y^4 + z^4 + w^4 = 0,$$

the equations of the 24 chords of curvature, right lines on $F$, being now in fact

I. (Chords, meeting $O_2O_3$, $O_1O_4$).

$$\begin{cases} x = \pm \alpha w, \\ y = \pm \alpha z, \end{cases} \quad \begin{cases} x = \pm \alpha^3 w, \\ y = \pm \alpha^3 z, \end{cases}$$

II. (Chords, meeting $O_3O_1$, $O_2O_4$).

$$\begin{cases} y = \pm \alpha w, \\ z = \pm \alpha x, \end{cases} \quad \begin{cases} y = \pm \alpha^3 w, \\ z = \pm \alpha^3 x, \end{cases}$$

III. (Chords, meeting $O_1O_2$, $O_3O_4$).

$$\begin{cases} z = \pm \alpha w, \\ x = \pm \alpha y, \end{cases} \quad \begin{cases} z = \pm \alpha^3 w, \\ x = \pm \alpha^3 y. \end{cases}$$

From the preceding it follows immediately that the surface $F$ is a covariant of the curve $R$ we originally started with, that is to say, the quaternary quartic $F$ is a combinant of the two quadrics $S$ and $S'$ we have used to define analytically the curve $R$. And indeed, assuming $S$ and $S'$ respectively to be $H_1 + \mu H_I$ and $H_1 + \mu' H_I$, and denoting by $U$ the reciprocal quadric of $S$ with respect to $S'$, and by $U'$ the reciprocal quadric of $S'$ with respect to $S$, there is no difficulty to show that $F$ is identical with

$$2US' - 2U'S - \theta' S^2 - \theta S'^2,$$

a quantic which may be readily recognized as a combinant of $S$ and $S'$.

Considering the two equations representing a biquadratic $R$ on $F$, we see that two such curves cannot have a point in common, an arbitrary quadric however, described through $R$, intersects the surface $F$ still in a second biquadratic $R'$, who of course meets $R$ in eight points. Hence there exists upon $F$ a second system of biquadratics $R'$, such that a quadric containing a curve $R$ also contains a curve $R'$, and vice versa; or, what is the same thing, every chord of a curve $R$ is at the same time a chord of one of the curves $R'$. This result can be established analytically. If the biquadratic $R$ of the first system be given by the equations

$$\begin{cases} \lambda(x^2 - iw^2) + \quad (y^2 - iz^2) = 0, \\ (x^2 + iw^2) - \lambda(y^2 + iz^2) = 0, \end{cases}$$

then we have only to alter the sign of $iw^2$ and to write $\mu$ for $\lambda$ to obtain a second biquadratic $R'$,

$$\begin{cases} \mu(x^2 + iw^2) + \quad (y^2 - iz^2) = 0, \\ (x^2 - iw^2) - \mu(y^2 + iz^2) = 0, \end{cases}$$

also situated on the tetrahedral quartic surface, and there is no difficulty in seeing that through both curves we can describe the quadric

$$f = x^2(\lambda + \mu) + y^2(1 - \lambda\mu) - iz^2(1 + \mu\lambda) + iw^2(\mu - \lambda) = 0.$$

Now if we try to find the coordinates of the eight points $P$, in which the curves $R$ and $R'$ meet, we get the four equations

$$x^2 = \lambda + \mu, \quad y^2 = 1 - \lambda\mu, \quad z^2 = -i(1 + \mu\lambda), \quad w^2 = i(\mu - \lambda),$$

and so it becomes evident that the quadric $f$ whereon both curves lie, is the polar quadric surface, with regard to $F$, of any one of these eight points $P$. Hence, since the tangents in $P$ to the curve of intersection of $F$ and this polar quadric are the two inflexional tangents in $P$, the two systems of biquadratics $R$ and $R'$ are proved to be identical with the asymptotic curves of the tetrahedral quartic surface $F$.

In drawing this conclusion we have arrived at a known result,* but at the same time the foregoing deductions allow us to add something to it, viz. the theorem that the tetrahedral quartic surface is completely determined as soon as one of its asymptotic curves is given.

And indeed if we start with the biquadratic $R$ and describe through $R$ an arbitrary quadric $f$, we can easily determine on $f$ the biquadratic $R'$ of the other system, it being sufficient to observe that in their meeting-points $P$, each of the curves $R$ and $R'$ touches one of the right lines of the quadric $f$.

* This result was first obtained by Lie ("Ueber die Reciprocitäts-Verhältnisse des Reye'schen Complexes"; Göttinger Nachrichten, 1870, p. 58.)

Let us next consider the right line, who joins two points $P$ and $Q$ on the biquadratic $R$. As it meets the surface $F$ in two points such that the polar quadric of either point passes through the other, we deduce from the ordinary theory of poles and polars that the line is cut harmonically in the two points $P$ and $Q$ and in the two remaining points where it meets the quartic surface again. All right lines, cut harmonically by the surface $F$, obviously constitute a complex of the sixth degree, and as the two cones, projecting from their common point $P$ the two biquadratics $R$ and $R'$, make up together a cone of that degree, we may enunciate the theorem: The system of the chords of the asymptotic curves on the tetrahedral quartic surface is identical with the complex of lines, meeting the surface in two pairs of harmonically conjugate points.

Having thus noticed the most important properties of the systems of biquadratics on $F$, there remains to observe that, as well as the curves $R$, the curves $R'$ have their chords of curvature in common. These 24 new right lines, completing the set of 48 right lines on $F$, will be found to be

I.
(Chords, meeting $O_2O_3$, $O_1O_4$).

$$\begin{cases} x = \pm\, aw \\ y = \pm\, a^3 z \end{cases}, \quad \begin{cases} x = \pm\, a^3 w \\ y = \pm\, az \end{cases},$$

II.
(Chords, meeting $O_3O_1$, $O_2O_4$).

$$\begin{cases} y = \pm\, aw \\ z = \pm\, a^3 x \end{cases}, \quad \begin{cases} y = \pm\, a^3 w \\ z = \pm\, ax \end{cases},$$

III.
(Chords, meeting $O_1O_2$, $O_3O_4$).

$$\begin{cases} z = \pm\, aw \\ x = \pm\, a^3 y \end{cases}, \quad \begin{cases} z = \pm\, a^3 w \\ x = \pm\, ay \end{cases},$$

and from these equations we readily can get an idea of the position of these 24 lines with respect to those of the other system. So, for example, is it easily verified that the lines $AF$, $FC$, $CH$, $HA$ and $EB$, $BG$, $GD$, $DE$, the sides of the two skew quadrilaterals $AFCH$ and $EBGD$, if drawn in the diagram above, would represent the eight chords of the first group. Moreover, it becomes now plain that we should consider the sides of any one of these skew quadrilaterals, formed by four chords of curvature, as a broken-up biquadratic $R$ or $R'$, each of the two systems $R$ and $R'$ containing six of these singular curves.

We conclude with some remarks about the invariants of the biquadratics on the surface $F$. A biquadratic is known to possess only one absolute invariant $A$, and if this invariant has the same value for two biquadratics, they can be transformed, one into another, by a linear substitution of the coordinates. In this case we shall say that the two curves are of the same *type*, and we will now ask to find on $F$ all the biquadratics of the same type as a given one $R$ represented by the equations

$$\begin{cases} \lambda(x^2 - iw^2) + \quad (y^2 - iz^2) = 0, \\ (x^2 + iw^2) - \lambda(y^2 + iz^2) = 0. \end{cases}$$

These two quadric surfaces determine a binary quartic

$$\Delta'\xi^4 + \theta'\xi^3 + \phi\xi^2 + \theta\xi + \Delta,$$

the invariants $I$ and $J$ of which we have to combine in order to find $I^3 : 16J^2$ as the expression of the absolute invariant $A$, belonging to the considered biquadratic. Working out the necessary calculations, the invariant $A$ of the curve $R$ with the parameter $\lambda$ becomes

$$A = \frac{(\lambda^8 + 14\lambda^4 + 1)^3}{(\lambda^{12} - 33\lambda^8 - 33\lambda^4 + 1)^2}.$$

In this expression we recognize the well-known forms of the theory of the octahedron, and remembering the 24 so-called octahedral substitutions, we infer that we have a set of 24 curves $R$ of the same type, the 24 corresponding values of the parameter $\lambda$ being

$$e^{\frac{i\pi k}{2}}\lambda, \quad e^{\frac{i\pi k}{2}} \times \frac{1}{\lambda}, \quad e^{\frac{i\pi k}{2}}\left(\frac{1+\lambda}{1-\lambda}\right), \quad e^{\frac{i\pi k}{2}}\left(\frac{1-\lambda}{1+\lambda}\right), \quad e^{\frac{i\pi k}{2}}\left(\frac{i+\lambda}{i-\lambda}\right), \quad e^{\frac{i\pi k}{2}}\left(\frac{i-\lambda}{i+\lambda}\right)$$
$$(k = 0,\ 1,\ 2,\ 3).$$

Quite the same result is arrived at when we consider the biquadratics $R'$ of the second system

$$\begin{cases} \mu(x^2 + iw^2) + \quad (y^2 - iz^2) = 0, \\ (x^2 - iw^2) - \mu(y^2 + iz^2) = 0. \end{cases}$$

Giving the parameter $\mu$ one of the 24 foregoing values, we obtain a further set of 24 curves $R'$, all of the same type as the curve $R$ with the parameter $\lambda$.

It thus being proved that there are in all 48 curves of a given type, we will now try to give some indications about their very symmetrical arrangement all over the surface $F$. In the first place we seek the locus of the meeting-points of a curve $R$ and a curve $R'$ of the same type, that is to say, we eliminate $\lambda$ and $\mu$ between the equations

$$x^2 = \lambda + \mu, \quad y^2 = 1 - \lambda\mu, \quad z^2 = -i(1 + \mu\lambda), \quad w^2 = i(\mu - \lambda)$$

in the understanding that $\lambda$ and $\mu$ are connected by one of the 24 octahedral substitutions. Considering the several cases, one after another, we shall find that the required locus is made up by the 24 curves of intersection of the surface $F$ with the following set of 24 quadrics $Q$, arranged in six groups of four:

$$\begin{cases} x^2 = 0, \\ y^2 = 0, \\ z^2 = 0, \\ w^2 = 0, \end{cases} \quad \begin{cases} x^2 + w^2 = 0, \\ x^2 - w^2 = 0, \\ y^2 + z^2 = 0, \\ y^2 - z^2 = 0, \end{cases} \quad \begin{cases} y^2 + w^2 = 0, \\ y^2 - w^2 = 0, \\ z^2 + x^2 = 0, \\ z^2 - x^2 = 0, \end{cases} \quad \begin{cases} z^2 + w^2 = 0, \\ z^2 - w^2 = 0, \\ x^2 - y^2 = 0, \\ x^2 + y^2 = 0, \end{cases}$$

$$\begin{cases} x^2 + y^2 + z^2 + w^2 = 0, \\ x^2 - y^2 - z^2 + w^2 = 0, \\ -x^2 + y^2 - z^2 + w^2 = 0, \\ -x^2 - y^2 + z^2 + w^2 = 0, \end{cases} \quad \begin{cases} -x^2 + y^2 + z^2 + w^2 = 0, \\ x^2 - y^2 + z^2 + w^2 = 0, \\ x^2 + y^2 - z^2 + w^2 = 0, \\ x^2 + y^2 + z^2 - w^2 = 0. \end{cases}$$

Now we can, if we like, define these quadrics as geometrical loci. For it is possible to establish a correspondence between the six Vossian quadrics of a curve $R$ and those belonging to a curve $R'$ of the same type in such a manner that by moving the curve $R$ along the tetrahedral quartic surface, the curve of intersection of each corresponding pair of Vossian quadrics describes one and the same quadric $Q$. There is however a more direct way to determine these quadrics $Q$ if we only consider those points on the surface $F$ where two right lines of different systems cross. Such points we have met already on the edges of the self-conjugate tetrahedron. In fact there are on each edge four points, where two chords of curvature of the biquadratics $R$ cross two chords of curvature of the curves $R'$.

Counting these points, as is naturally, four times as a meeting-point of two right lines of different systems, and combining them with the 192 crossing-points found elsewhere on the surface, we obtain a total number of 288 of these special points. And by the help of them we can determine the quadrics $Q$, as any one of these surfaces passes through 48 of these points.

If, then, in this way these quadrics $Q$ have been constructed, we can get by their aid some idea of the arrangement of the set of 48 biquadratics of a given type; for in fact these quadrics $Q$ in some sort link together in their eight common points each pair of curves $R$ and $R'$. And this arrangement becomes more simple and less intricate by considering the already indicated division of the quadrics $Q$ into six groups of four. Starting with a given curve $R$ and seeking for all biquadratics on $F$ that can be linked to $R$, directly or indirectly, by means of the four quadrics $Q$ of one group, we shall come to the conclusion that no more than four curves of each system can be found. So then we get a perfectly regular configuration consisting of four curves $R$ and four curves $R'$, all of the same type, any pair of curves of different systems meeting each other on one of the four quadrics $Q$.

Again, taking in particular the first group of these quadrics $Q$, we see that, starting with any curve $R$, we can construct a special configuration in which each pair of consecutive curves is linked together by one of the faces of the self-conjugate tetrahedron, the two curves touching each other in four distinct points.

A final remark may be added about the linear substitutions of the coordinates by which the tetrahedral quartic surface $F$ is transformed into itself. It is possible to show that the homographic transformation, which has the effect of changing a curve $R$ with the parameter $\lambda$ into one of the 47 other curves of the same type, does not involve the constant $\lambda$, and therefore by each of these transformations the surface $F$ is transformed into itself. Now by introducing the elliptic argument, as we have done before, it is immediately apparent that a biquadratic can be transformed into itself in 32 different ways. Hence we are justified in stating that there are $32 \times 48 = 1536$ distinct linear substitutions of the coordinates which do not alter the equation of the surface $F$, a statement already made by Schur.*

* "Ueber Flächer vierter Ordnung," Math. Annalen, Bd. 20, p. 294.

# *Calcul Géométrique Réglé.*

PAR RENÉ DE SAUSSURE.

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Dans un article précédent, intitulé "Etude de géométrie cinématique réglée,"* j'ai montré comment l'on peut étudier la géométrie de la droite en considérant l'espace réglé comme la représentation de la surface ponctuelle d'une sphère imaginaire de rayon $i=\sqrt{-1}$. La méthode exposée dans cet article était purement géométrique, c'est-à-dire d'un emploi relativement restreint; je me propose de reprendre le même sujet au point de vue analytique, ce qui m'amènera à établir les règles d'un nouveau calcul géométrique et à en montrer les principales applications.†

Je diviserai le sujet en cinq parties:

## I°.—*Règles de Calcul.*

Rappelons d'abord en peu de mots le principe qui nous a servi de base pour l'étude géométrique de l'espace réglé; on considère une sphère imaginaire fondamentale de rayon $i$ et l'on fait correspondre à chaque point de sa surface une droite *réelle* de l'espace, ce qui est possible, puisqu'une surface imaginaire contient une quadruple infinité de points; il en résulte qu'à toute figure composée de points sur la sphère correspond une figure composée de droites dans l'espace et s'il existe certaines relations entre les points de la figure sphérique, les mêmes relations existeront entre les droites de l'espace, de sorte que l'on peut déduire la géométrie réglée de la géométrie sphérique.

En effet, deux droites $A$ et $B$ dans l'espace déterminent une grandeur complexe de la forme $P+QI$, $P$ étant la plus courte distance des deux droites, $Q$ leur angle et $I$ un symbole unité; cette grandeur complexe que nous avons

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* Voir vol. XVIII, No. 4.

† Les premiers principes de ce calcul ont été exposés dans les Comptes Rendus (Séances du 9 et du 16 Novembre 1896).

appelée *distangle* des droites $A$ et $B$ (par abréviation pour distance-angle), correspond sur la sphère à la longueur $p+qi$ de l'arc de grand cercle imaginaire déterminé par deux points $a$ et $b$. L'arc $p+qi$ est la mesure linéaire de la distance des points $a$ et $b$, tandis que l'expression $\frac{p+qi}{i}$ est la mesure angulaire de cette même distance, puisque $i$ est le rayon de la sphère; on doit donc considérer le distangle $P+QI$ formé par deux droites $A$ et $B$ comme la mesure *linéaire* de l'intervalle compris entre ces droites, tandis qu'on prendra pour mesure *angulaire* du même intervalle l'expression $\frac{P+QI}{I}$, qui est bien homogène et de degré nul, si l'on regarde le symbole $I$ comme l'équivalent d'une longueur. Cette mesure angulaire, que nous avons nommée *codistangle*, sera désignée par le symbole $(\overline{AB})$, de sorte qu'on a par définition:

$$(\overline{AB}) = \frac{P+QI}{I}.$$

Le codistangle est la grandeur fondamentale de l'espace réglé, comme l'angle est celle de la géométrie sphérique; mais, pour qu'il y ait parfaite identité entre la géométrie réglée et la géométrie sur une sphère imaginaire, il est nécessaire de soumettre les codistangles à des règles de calcul s'écartant un peu des règles ordinaires, parce que la constitution de l'espace réglé n'est pas absolument identique à celle de la sphère imaginaire ainsi que nous le verrons bientôt.

Lorsqu'on opère avec des angles imaginaires de la forme $\frac{p+qi}{i}$, on traite le symbole $i$ comme une lettre ordinaire et l'on pose $i^2 = -1$, de sorte que le résultat est toujours réductible à un angle de la même forme. Si l'on opère avec des codistangles, rien n'autorise à poser $I^2 = -1$, mais on peut présumer que toute fonction d'un codistangle sera un nouveau codistangle:

$$F\left(\frac{x_1 + Ix_2}{I}\right) = \frac{y_1 + Iy_2}{I},$$

$x_1$ et $y_1$ étant des longueurs, $x_2$ et $y_2$ des angles. Cette équation complexe doit être équivalente à deux équations ordinaires entre les quantités $x_1$, $y_1$, $x_2$, $y_2$; on a en effet, d'après la formule de Taylor:

$$F\left(\frac{x_1 + Ix_2}{I}\right) = F\left(x_2 + \frac{x_1}{I}\right) = F(x_2) + \frac{x_1}{I}\,\frac{dF(x_2)}{dx_2} + \text{etc.}\dots,$$

c'est-à-dire :

$$F\left(\frac{x_1+Ix_2}{I}\right)=\frac{x_1\dfrac{dF(x_2)}{dx_2}+IF(x_2)}{I}+\text{etc.}\ldots\ldots$$

Le premier terme du second membre est précisément un codistangle de la forme $\frac{y_1+Iy_2}{I}$ ; il n'y a pas lieu de s'occuper des autres termes du développement, car ces termes sont d'une nature essentiellement différente et irréductible, puisqu'ils contiennent des puissances de $I$ au dénominateur. On rejettera donc ces termes comme n'ayant pas de sens dans l'équation. C'est ainsi qu'on aura :

$$\left\{\begin{aligned}
\sin\left(\frac{x_1+Ix_2}{I}\right)&=\frac{x_1\cos x_2+I\sin x_2}{I}, &(1)\\
\cos\left(\frac{x_1+Ix_2}{I}\right)&=\frac{-x_1\sin x_2+I\cos x_2}{I}, &(2)\\
\operatorname{tang}\left(\frac{x_1+Ix_2}{I}\right)&=\frac{\dfrac{x_1}{\cos^2 x_2}+I\operatorname{tang}x_2}{I}. &(3)
\end{aligned}\right.$$

En résumé, on voit que le symbole $I$ peut être traité comme une lettre ordinaire dans tous les calculs, pourvu qu'après chaque opération, l'on supprime les termes qui ne représentent pas des codistangles. Les règles fondamentales de calcul seront donc les suivantes :

$$\frac{a_1+Ia_2}{I}+\frac{b_1+Ib_2}{I}=\frac{(a_1+b_1)+I(a_2+b_2)}{I},$$

$$\left(\frac{a_1+Ia_2}{I}\right)\left(\frac{b_1+Ib_2}{I}\right)=\frac{(a_1b_2+a_2b_1)+I(a_2b_2)}{I}, \qquad (4)$$

$$\frac{a_1+Ia_2}{b_1+Ib_2}=\frac{\left(\dfrac{a_1b_2-a_2b_1}{b_2^2}\right)+I\left(\dfrac{a_2}{b_2}\right)}{I},$$

$$\left(\frac{a_1+Ia_2}{I}\right)^m=\frac{ma_1a_2^{m-1}+Ia_2^m}{I}, \qquad (5)$$

$$d\left(\frac{x_1+Ix_2}{I}\right)=\frac{dx_1+Idx_2}{I},$$

$$\int\left(\frac{y_1+Iy_2}{I}\right)d\left(\frac{x_1+Ix_2}{I}\right)=\frac{\int(y_1\,dx_2+y_2\,dx_1)+I\int y_2\,dx_2}{I}.$$

Il est facile de voir qu'avec ces règles spéciales de calcul, toutes les formules ordinaires d'analyse subsistent, lorsque les lettres qui entrent dans la formule représentent des codistangles. Soit par exemple: $a = \frac{a_1 + Ia_2}{I}$, $b = \frac{b_1 + Ib_2}{I}$, etc., on vérifiera sans difficulté que:

$$\begin{cases} \sin^2 a + \cos^2 a = 1, \\ \cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \text{etc.}, \end{cases} \qquad \begin{cases} d \sin x = \cos x \, dx, \\ d(ab) = adb + bda \end{cases}$$

et ainsi de suite. Il en résulte que toute relation entre les angles d'une figure sphérique subsistera aussi entre les codistangles de la figure correspondante dans l'espace réglé. C'est ainsi que la trigonométrie sphérique peut être généralisée de manière à constituer une trigonométrie de l'espace réglé.

## II°.—*Trigonométrie réglée.*

Trois points $x$, $y$, $z$ déterminent sur la sphère un triangle sphérique; trois droites $X$, $Y$, $Z$ dans l'espace déterminent une figure que nous avons appelée *tridistangle*. Les trois codistangles $(\overline{YZ})$, $(\overline{ZX})$, $(\overline{XY})$, correspondent aux trois côtés du triangle sphérique. Si le triangle $xyz$ est trirectangle, la figure correspondante $XYZ$ sera un trièdre trirectangle, car les trois côtés d'un tel tridistangle sont bien égaux à $\frac{\pi}{2}$, puisque dans ce cas:

$$(\overline{YZ}) = (\overline{ZX}) = (\overline{XY}) = \frac{0 + I\frac{\pi}{2}}{I} = \frac{\pi}{2}.$$

Si maintenant $m$ est un point quelconque de la sphère, on sait que:

$$\cos^2(mx) + \cos^2(my) + \cos^2(mz) = 1.$$

Cette formule de trigonométrie sphérique doit subsister dans l'espace réglé, c'est-à-dire que, si $XYZ$ est un trièdre trirectangle (ou si l'on veut un système d'axes de coordonnées rectangulaires) et $M$ une droite quelconque de l'espace, on doit avoir:

$$\cos^2(\overline{MX}) + \cos^2(\overline{MY}) + \cos^2(\overline{MZ}) = 1. \tag{6}$$

Cette relation constitue une formule de trigonométrie réglée et pour l'interpréter, il suffit d'identifier les deux membres après avoir ramené chacun d'eux à la forme normale $\frac{P+QI}{I}$, au moyen des règles de calcul précédemment établies. Soient donc $\alpha_1$, $\beta_1$, $\gamma_1$, les plus courtes distances et $\alpha_2$, $\beta_2$, $\gamma_2$, les angles compris entre la droite $M$ et chacun des axes $X$, $Y$, $Z$; la formule devient:

$$\cos^2\left(\frac{\alpha_1+I\alpha_2}{I}\right)+\cos^2\left(\frac{\beta_1+I\beta_2}{I}\right)+\cos^2\left(\frac{\gamma_1+I\gamma_2}{I}\right)=1,$$

ou plus simplement:

$$\sum\cos^2\left(\frac{\alpha_1+I\alpha_2}{I}\right)=1.$$

La formule (2) permet de mettre cette équation sous la forme:

$$\sum\left[\frac{-\alpha_1\sin\alpha_2+I\cos\alpha_2}{I}\right]^2=1$$

et en effectuant les carrés, d'après la formule (5):

$$\sum\frac{-2\alpha_1\sin\alpha_2\cos\alpha_2+I\cos^2\alpha_2}{I}=1=\frac{0+I}{I}.$$

On a donc, en identifiant les deux membres:

$$\begin{cases}\cos^2\alpha_2+\cos^2\beta_2+\cos^2\gamma_2 & =1,\\ \alpha_1\sin 2\alpha_2+\beta_1\sin 2\beta_2+\gamma_1\sin 2\gamma_2 & =0.\end{cases}$$

C'est ainsi que toute formule de trigonométrie sphérique conduit à une formule identique de trigonométrie réglée, d'où l'on déduit *deux* formules: la première n'est pas autre chose que celle qui a servi de point de départ, parce qu'elle n'implique que la direction des droites; mais la seconde est une formule nouvelle, qui établit une relation non-seulement entre les angles, mais aussi entre les distances des droites considérées dans l'espace. Les nouvelles formules que l'on obtient ainsi ont toujours une forme aussi simple et aussi symétrique que possible et il serait souvent difficile, ou en tout cas assez long de les établir par les méthodes ordinaires de la géométrie analytique.

Nous n'avons considéré encore, soit dans les triangles sphériques soit dans les tridistangles, que trois éléments, savoir les trois côtés. La plupart des formules de trigonométrie sphérique contiennent en outre les angles compris entre les côtés, mais on peut toujours remplacer ces angles par les côtés du triangle polaire. De même, tout tridistangle $ABC$ contient, outre ses trois côtés $(\overline{BC})$, $(\overline{CA})$, $(\overline{AB})$, trois éléments qui ne sont autre chose que les côtés du tridistangle polaire $XYZ$, formé par les trois perpendiculaires communes aux droites $A$, $B$ et $C$ prises deux à deux.* Toute relation entre les côtés et les angles d'un triangle sphérique, existant aussi entre les éléments correspondants d'un tridistangle, on aura par exemple:

$$\cos(\overline{BC}) = \cos(\overline{CA})\cos(\overline{AB}) + \sin(\overline{CA})\sin(\overline{AB})\cos(\overline{YZ}).$$

En opérant comme précédemment, c'est-à-dire en posant:

$$(\overline{BC}) = \frac{a_1 + Ia_2}{I}, \quad (\overline{YZ}) = \frac{x_1 + Ix_2}{I}$$

et permutant circulairement, on trouvera en identifiant les deux membres de la formule après les avoir ramenés à la forme normale:

$$\begin{cases} \cos a_2 = \cos b_2 \cos c_2 + \sin b_2 \sin c_2 \cos x_2, \\ a_1 \sin a_2 = (b_1 - c_1 \cos x_2) \sin b_2 \cos c_2 + (c_1 - b_1 \cos x_2) \sin c_2 \cos b_2 \\ \qquad\qquad + x_1 \sin x_2 \sin b_2 \sin c_2. \end{cases}$$

Dans le cas particulier où le codistangle $(\overline{YZ})$ est égal à $\frac{\pi}{2}$, on dit que le tridistangle $ABC$ est rectangle le long de la droite $A$; pour que cela ait lieu, il faut que $x_1 = 0$ et $x_2 = \frac{\pi}{2}$, c'est-à-dire que les trois droites $Y$, $Z$ et $A$ forment un trièdre trirectangle; dans ce cas, on a simplement:

$$\cos(\overline{BC}) = \cos(\overline{CA})\cos(\overline{AB}), \tag{7}$$

c'est-à-dire:

$$\begin{cases} \cos a_2 = \cos b_2 \cos c_2, \\ a_1 \sin a_2 = b_1 \sin b_2 \cos c_2 + c_1 \sin c_2 \cos b_2. \end{cases}$$

Lorsque les droites $A$, $B$, $C$ sont infiniment voisines l'une de l'autre, on peut développer les cosinus de la formule (7) en série et négliger les termes d'ordre supérieur, ce qui donne:

$$\overline{d(\overline{BC})}^2 = \overline{d(\overline{CA})}^2 + \overline{d(\overline{AB})}^2.$$

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* Voir vol. XVIII, p. 311.

C'est le théorème de Pythagore pour l'espace réglé, théorème qui est équivalent aux deux relations:

$$\begin{cases} da_2^2 = db_2^2 + dc_2^2, \\ da_1\,da_2 = db_1\,db_2 + dc_1\,dc_2. \end{cases}$$

Enfin dans tout tridistangle rectangle infinitésimal on aura aussi:

$$\begin{cases} \sin\left(\dfrac{y_1 + Iy_2}{I}\right) = \dfrac{db_1 + Idb_2}{da_1 + Ida_2}, \\ \cos\left(\dfrac{y_1 + Iy_2}{I}\right) = \dfrac{dc_1 + Idc_2}{da_1 + Ida_2}, \\ \operatorname{tang}\left(\dfrac{y_1 + Iy_2}{I}\right) = \dfrac{db_1 + Idb_2}{dc_1 + Idc_2}, \end{cases} \qquad (8)$$

c'est-à-dire que par exemple le sinus du codistangle appuyé sur la droite $B$ est égal au rapport du côté opposé $(\overline{CA})$ à l'hypoténuse $(\overline{BC})$, etc. La définition géométrique des fonctions circulaires est ainsi étendue aux fonctions trigonométriques d'un codistangle. Du reste on tirerait aussi des formules (8), en effectuant les divisions dans les seconds membres:

$$\begin{cases} \sin y_2 = \dfrac{db_2}{da_2}, \\ \cos y_2 = \dfrac{dc_2}{da_2}, \\ \operatorname{tang} y_2 = \dfrac{db_2}{dc_2}, \end{cases} \qquad \begin{cases} y_1 \cos y_2 = \dfrac{db_1\,da_2 - da_1\,db_2}{da_2^2}, \\ -\,y_1 \sin y_2 = \dfrac{dc_1\,da_2 - da_1\,dc_2}{da_2^2}, \\ \dfrac{y_1}{\cos^2 y_2} = \dfrac{db_1\,dc_2 - dc_1\,db_2}{dc_2^2}. \end{cases}$$

### III°.—*Géométrie réglée synthétique.*

Toutes les figures sphériques que nous considérons se composent de points et de lignes situés sur la surface même de la sphère fondamentale. Les figures réglées qui leur correspondent se composent donc de droites et de congruences de droites dans l'espace, car toute courbe sphérique imaginaire contient une double infinité de points.

C'est ainsi que la congruence qui correspond à un grand cercle de la sphère et que nous avons appelée *recticongruence*, est formée de toutes les droites qui rencontrent une droite donnée $P$ sous un angle droit; la droite $P$ est le pôle de la congruence car elle fait avec une droite quelconque du lieu un codistangle égal à $\dfrac{0 + I\frac{\pi}{2}}{I} = \dfrac{\pi}{2}$. On voit que les recticongruences sont les géodésiques de l'espace réglé.

La *congruence circulaire*, c'est-à-dire celle qui correspond à un petit cercle de la sphère, est formée de toutes les droites tangentes à un cylindre de révolution et également inclinées sur les génératrices de celui-ci ; l'axe du cylindre est le pôle de la congruence, car le codistangle compris entre cet axe et une droite quelconque du lieu est constant.

On définira de même la *congruence elliptique* comme le lieu des droites qui forment avec deux droites données, des codistangles dont la somme est constante, et ainsi de suite. Nous avons montré que toutes les congruences qui correspondent ainsi à des courbes sphériques, sont formées de normales à une surface développable ; ces congruences sont dites *analytiques*.* Nous avons aussi défini la *tangente* à une congruence le long d'une de ses génératrices $D$, comme étant la recticongruence passant par $D$ et par une génératrice infiniment voisine $D'$ ; le pôle $T$ de la tangente est donc la perpendiculaire commune à $D$ et à $D'$ ; ce pôle est d'ailleurs le même quelle que soit la génératrice voisine que l'on considère, pourvu que la congruence soit analytique. De même, la *normale* à la congruence le long de $D$ est la recticongruence perpendiculaire à la tangente ; le pôle $N$ de cette normale forme donc avec les droites $D$ et $T$ un trièdre trirectangle. La *congruence circulaire osculatrice* le long de $D$ est celle qui est déterminée par $D$ et deux autres génératrices voisines (n'appartenant pas à la même tangente). Le pôle $K$ de cette congruence circulaire est situé sur la normale ; c'est l'*axe de courbure* de la congruence et le codistangle $(\overline{KD})$ est le *rayon de courbure*. Enfin, étant données deux génératrices $A$ et $B$ d'une congruence analytique, on entend par *arc* $AB$ la somme des codistangles élémentaires engendrés par une droite qui passerait de la position $A$ à la position $B$ sans quitter la congruence ; cette somme est indépendante du chemin suivi entre $A$ et $B$.

Les formules relatives aux arcs de courbes sphériques subsisteront évidemment pour les arcs de congruence. Ainsi par exemple, un arc de cercle $s$ tracé

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* Voir vol. XVIII, p. 314.

sur la sphère avec un rayon angulaire $r$ et correspondant à un angle au centre $\omega$, a pour valeur:

$$s = \omega \sin r.$$

De même, si $A$ et $B$ sont deux génératrices d'une congruence circulaire dont le pôle est $X$ (Fig. 1), le rayon de la congruence sera $r = \frac{r_1 + Ir_2}{I}$, $r_1$ étant le rayon

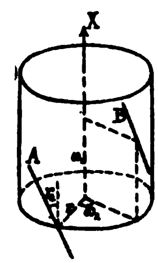


FIG. 1.

du cylindre focal et $r_2$ l'inclinaison de $A$ ou $B$ sur les génératrices de ce cylindre; le codistangle au centre correspondant à l'arc $AB$ sera $\omega = \frac{\omega_1 + I\omega_2}{I}$, $\omega_2$ étant l'angle formé par les normales au cylindre élevées aux points de contact des droites $A$ et $B$ et $\omega_1$ le segment intercepté par ces normales sur l'axe du cylindre. Si donc une droite passe de la position $A$ à la position $B$ en restant tangente au cylindre et en faisant constamment le même angle avec les génératrices, la somme $s = \frac{s_1 + Is_2}{I}$ des codistangles élémentaires engendrés par cette droite ne dépendra pas du chemin suivi et l'on aura comme sur la sphère:

$$\frac{s_1 + Is_2}{I} = \left(\frac{\omega_1 + I\omega_2}{I}\right) \sin \left(\frac{r_1 + Ir_2}{I}\right),$$

c'est-à-dire, en identifiant les deux membres:

$$\begin{cases} s_2 = \omega_2 \sin r_2, \\ s_1 = \omega_1 \sin r_2 + r_1\omega_2 \cos r_2, \end{cases}$$

valeurs que l'on peut vérifier en effectuant directement les sommes $s_1$ et $s_2$ entre $A$ et $B$.

Le principe de dualité, si utile en géométrie sphérique, existe aussi dans l'espace réglé, car à toute propriété d'une figure composée de droites correspond une propriété corrélative de la figure polaire. Ainsi on sait qu'un quadrilatère sphérique est inscriptible dans un cercle lorsque la somme de deux angles opposés est égale à la somme des deux autres angles, et l'on en déduit au moyen de la figure polaire qu'un quadrilatère sphérique est circonscriptible lorsque la somme de deux côtés opposés est égale à la somme des deux autres côtés.

De même, quatre droites $A$, $B$, $C$, $D$ dans l'espace déterminent une figure ayant quatre côtés $(\overline{AB})$, $(\overline{BC})$, $(\overline{CD})$ et $(\overline{DA})$; si les droites $P$, $Q$, $R$, $S$ qui forment les pôles de ces côtés satisfont à la condition:

$$(\overline{SP}) + (\overline{QR}) = (\overline{PQ}) + (\overline{RS}),$$

les droites $A$, $B$, $C$, $D$ appartiendront à une même congruence circulaire, c'est-à-dire qu'il existera un cylindre de révolution tangent à ces quatre droites et dont l'axe est également incliné sur chacune d'elles.

En posant:

$$(\overline{SP}) = \frac{a_1 + Ia_2}{I}, \quad (\overline{PQ}) = \frac{b_1 + Ib_2}{I}, \quad (\overline{QR}) = \frac{c_1 + Ic_2}{I}, \quad (\overline{RS}) = \frac{d_1 + Id_2}{I},$$

la relation précédente devient:

$$\begin{cases} a_1 + c_1 = b_1 + d_1, \\ a_2 + c_2 = b_2 + d_2. \end{cases}$$

Ainsi ces deux égalités expriment, soit que la figure $ABCD$ est inscrite dans une congruence circulaire, soit que la figure $PQRS$ est circonscrite à une congruence de même nature.

Pour exprimer que quatre droites $PQRS$ forment une figure inscriptible, on peut aussi se servir du théorème de Ptolémée (pour les quadrilatères sphériques):

$$\sin\left(\frac{\overline{PR}}{2}\right)\sin\left(\frac{\overline{QS}}{2}\right) = \sin\left(\frac{\overline{SP}}{2}\right)\sin\left(\frac{\overline{QR}}{2}\right) + \sin\left(\frac{\overline{PQ}}{2}\right)\sin\left(\frac{\overline{RS}}{2}\right).$$

En posant: $(\overline{PR}) = \frac{x_1 + Ix_2}{I}$ et $(\overline{QS}) = \frac{y_1 + Iy_2}{I}$, ce théorème fournit les deux relations suivantes entre les angles et les distances des quatre droites:

$$\begin{cases} \sin\frac{x_2}{2}\sin\frac{y_2}{2} = \sin\frac{a_2}{2}\sin\frac{c_2}{2} + \sin\frac{b_2}{2}\sin\frac{d_2}{2}, \\ x_1\cos\frac{x_2}{2}\sin\frac{y_2}{2} + y_1\cos\frac{y_2}{2}\sin\frac{x_2}{2} = a_1\cos\frac{a_2}{2}\sin\frac{c_2}{2} \\ \qquad + b_1\cos\frac{b_2}{2}\sin\frac{d_2}{2} + c_1\cos\frac{c_2}{2}\sin\frac{a_2}{2} + d_1\cos\frac{d_2}{2}\sin\frac{b_2}{2}. \end{cases}$$

Du reste il existe toujours entre quatre droites quelconques $P$, $Q$, $R$, $S$ une relation identique à celle qui existe entre quatre points quelconques d'une sphère ; en appelant $\alpha, \beta, \gamma, \delta, \xi, \eta$ les cosinus des quatre côtés $(\overline{SP})$, $(\overline{PQ})$, $(\overline{QR})$, $(\overline{RS})$ et des deux diagonales $(\overline{QS})$, $(\overline{PR})$, on a donc :

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\xi^2+\eta^2-\xi^2\eta^2-\alpha^2\gamma^2-\beta^2\delta^2$$
$$-2\alpha\beta\xi-2\gamma\delta\xi-2\alpha\delta\eta-2\beta\gamma\eta+2\alpha\gamma\xi\eta+2\beta\delta\xi\eta+2\alpha\beta\gamma\delta=1.$$

Cette équation complexe fournira deux relations entre les angles et les distances des droites.

Pour faire passer une congruence circulaire par trois droites données $A$, $B$, $C$, il suffit d'en déterminer le rayon $R=\frac{R_1+IR_2}{I}$. Dans ce but, on pourra employer toute formule donnant le rayon du cercle circonscrit à un triangle sphérique $ABC$. On aura par exemple, en employant les mêmes notations qu'en trigonométrie réglée :

$$\operatorname{tang} R=\frac{\operatorname{tang}\frac{1}{2}(\overline{BC})}{\cos[S-(\overline{YZ})]} \qquad \text{avec } S=\frac{(\overline{YZ})+(\overline{ZX})+(\overline{XY})}{2}$$

et l'on en déduit par la méthode ordinaire :

$$\begin{cases} \operatorname{tang} R_2=\dfrac{\operatorname{tang}(\frac{1}{2}a_2)}{\cos(s_2-x_2)}, \\ R_1=\dfrac{\frac{1}{2}\dfrac{a_1}{\cos^2(\frac{1}{2}a_2)}\cos(s_2-x_2)+(s_1-x_1)\operatorname{tang}(\frac{1}{2}a_2)\sin(s_2-x_2)}{\cos^2(s_2-x_2)+\operatorname{tang}^2(\frac{1}{2}a_2)}. \end{cases}$$

*Des surfaces réglées.* Jusqu'ici, nous avons considéré la congruence, ou plus exactement la congruence analytique, comme la forme fondamentale de l'espace réglé. Mais la même méthode permettra d'étudier les surfaces réglées, en considérant celles-ci comme parties constituantes de congruences analytiques.

Deux génératrices voisines d'une surface réglée déterminent une recticongruence *tangente* à cette surface le long de la génératrice considérée ; la *normale* est la recticongruence perpendiculaire à la tangente ; trois génératrices infiniment voisines déterminent la congruence circulaire osculatrice ; le pôle de cette congruence est l'*axe de courbure* de la surface réglée et le codistangle compris entre cet axe et la génératrice en est le *rayon de courbure.*

Etant donnée une surface réglée quelconque, il existe toujours une congruence analytique et une seule contenant cette surface.* De plus, les éléments de courbure, de contact, etc. d'une surface réglée sont les mêmes que ceux de la congruence analytique qui contient cette surface; ceci résulte de la définition même de ces éléments.

Ainsi les formules de la géométrie sphérique s'appliqueront aux surfaces réglées aussi bien et même mieux qu'aux congruences, puisqu'il n'y a aucune restriction à faire sur la nature des surfaces réglées que l'on considère.

Cherchons par exemple l'expression du rayon de courbure d'une surface réglée. Soient $D$, $D'$, $D''$ trois génératrices consécutives de la surface, $X$ l'axe de courbure correspondant; $T$ et $T'$ les pôles des tangentes relatives aux génératrices $D$ et $D'$, c'est-à-dire que $T$ est la perpendiculaire commune à $D$ et $D'$ et $T'$ la perpendiculaire à $D'$ et $D''$. Le codistangle $(\overline{DD'}) = ds = \frac{ds_1 + Ids_2}{I}$ est alors l'élément d'arc de la surface réglée; $(\overline{TT'}) = d\omega = \frac{d\omega_1 + Id\omega_2}{I}$ est le *codistangle de contingence* et $(\overline{DX}) = r = \frac{r_1 + Ir_2}{I}$, le rayon de courbure de la surface. On a donc, comme pour les courbes sphériques:

$$\operatorname{tang} r = \frac{ds}{d\omega},$$

on en effectuant les calculs et identifiant les deux membres:

$$\begin{cases} \operatorname{tang} r_2 = \dfrac{ds_2}{d\omega_2}, \\ \qquad r_1 = \dfrac{ds_1\, d\omega_2 - d\omega_1\, ds_2}{d\omega_2^2 + ds_2^2}. \end{cases}$$

Ces valeurs de $r_1$ et de $r_2$ déterminent sur la normale la position de l'axe de courbure $X$.

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* Voir vol. XVIII, p. 324.

Les pôles $T$, $T'$, etc. des tangentes forment une surface réglée réciproque ou polaire de la surface $(D)$; cette surface polaire $(T)$ a même axe de courbure que $(D)$; en appelant $p$ le paramètre de distribution de la surface $(D)$ et $\pi$ celui de la surface polaire $(T)$, on a $p = \frac{ds_1}{ds_2}$, $\pi = \frac{d\omega_1}{d\omega_2}$ et l'on tire des équations précédentes la relation:

$$p - \pi = r_1 (\text{tang}\, r_2 + \text{cotang}\, r_2).$$

Nous reparlerons des surfaces réglées après avoir défini les différents systèmes de coordonnées dans l'espace réglé.

## IV°.—*Géométrie analytique réglée.*

On définira la position d'une droite dans l'espace, comme celle d'un point de la sphère imaginaire, au moyen de deux coordonnées complexes.

1°. *Coordonnées polaires.* On se donne le pôle $X$ d'une recticongruence fixe et l'on prend comme droite origine $O$ une certaine génératrice de cette congruence (Fig. 2). La position d'une droite quelconque $M$ sera définie si l'on définit

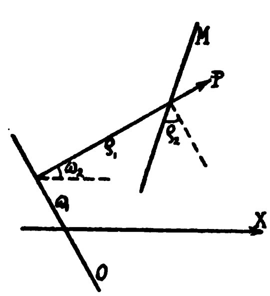



FIG. 2.

d'abord la position de la droite $P$ (perpendiculaire commune à $O$ et à $M$), au moyen du codistangle $(\overline{PX}) = \omega = \frac{\omega_1 + I\omega_2}{I}$; ensuite la position de la droite $M$ dans la recticongruence dont le pôle est $P$, au moyen du codistangle $(\overline{OM}) = \rho = \frac{\rho_1 + I\rho_2}{I}$.

Ce système correspond aux coordonnées polaires sur la sphère, car $(\overline{OM}) = \rho$ n'est pas autre chose que la *colatitude* de la droite $M$ et $(\overline{PX}) = \omega$ est bien la *longitude*, puisque ce codistangle mesure l'inclinaison des congruences $P$ et $X$ l'une sur l'autre. L'équation $\rho = \text{const.}$ représente une congruence circulaire dont le pôle est $O$, c'est-à-dire un *parallèle* de l'espace réglé, tandis que $\omega = \text{const.}$ représente un *méridien*.

Toute relation entre $\rho$ et $\omega$ représente une congruence analytique et cette relation est identique à l'équation en coordonnées polaires de la courbe sphérique correspondante.

2°. *Coordonnées tripolaires.* On définit souvent la position d'un point $m$ sur une sphère au moyen des distances sphériques de ce point à trois points fixes $x$, $y$, $z$, formant un triangle trirectangle; ces trois coordonnées $\alpha$, $\beta$, $\gamma$ du point $m$ ne sont pas indépendantes et l'on a: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Pour trouver la position du point $m$ lorsqu'on connaît ses coordonnées, on décrit des cercles ayant respectivement pour pôles les points $x$, $y$, $z$ et pour rayons les arcs $\alpha$, $\beta$, $\gamma$; ces trois cercles ont un seul point commun.

On peut définir de même la position d'une droite $M$ dans l'espace, au moyen des codistangles compris entre cette droite et trois axes fixes $X$, $Y$, $Z$ formant un trièdre trirectangle. Les coordonnées tripolaires de la droite $M$ sont ainsi:

$$(\overline{MX}) = \alpha = \frac{\alpha_1 + I\alpha_2}{I},$$

$$(\overline{MY}) = \beta = \frac{\beta_1 + I\beta_2}{I},$$

$$(\overline{MZ}) = \gamma = \frac{\gamma_1 + I\gamma_2}{I}$$

et l'on a encore: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Pour trouver la position de la droite $M$ lorsqu'on connaît ses coordonnées tripolaires, on décrira des congruences circulaires ayant respectivement pour pôles les droites $X$, $Y$, $Z$ et pour rayons les codistangles $\alpha$, $\beta$, $\gamma$; ces trois congruences ont une seule droite commune. Il est donc nécessaire de savoir déterminer géométriquement l'intersection de deux congruences circulaires dont on se donne les pôles $X$, $Y$ et les rayons $\alpha$, $\beta$: on décrit autour de la droite $X$ un cylindre de rayon $\alpha_1$ et autour de la droite $Y$, un cylindre de rayon $\beta_1$; la droite cherchée $M$ devra être tangente à ces deux cylindres; de plus, elle doit faire avec $X$ un angle égal à $\alpha_2$ et avec $Y$, un angle égal à $\beta_2$; la direction de $M$ est ainsi déterminée par l'intersection de deux cônes de même sommet dont les axes sont respectivement parallèles à $X$ et à $Y$ et dont les génératrices font avec ces axes les angles $\alpha_2$ et $\beta_2$. On mènera donc à chacun des deux cylindres un plan tangent parallèle à la direction trouvée et l'intersection de ces deux plans sera la droite cherchée $M$; il y a en général deux solutions.

Comme exemple de ce genre de coordonnées, cherchons la valeur de la plus courte distance $V_1$ et de l'angle $V_2$ compris entre deux droites $M$ et $M'$ dont les coordonnées tripolaires sont $\alpha$, $\beta$, $\gamma$ et $\alpha'$, $\beta'$, $\gamma'$. En posant $\frac{V_1 + IV_2}{I} = V$, on aura, comme pour la distance de deux points sur la sphère :

$$\cos V = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma',$$

ou plus simplement :

$$\cos V = \Sigma \cos\alpha \cos\alpha'.$$

En ramenant chaque membre de cette équation à la forme normale et identifiant, on aura :

$$\begin{cases} \cos V_2 = \Sigma \cos\alpha_2 \cos\alpha_2', \\ V_1 \sin V_2 = \Sigma (\alpha_1 \sin\alpha_2 \cos\alpha_2' + \alpha_1' \sin\alpha_2' \cos\alpha_2), \end{cases}$$

équations qui déterminent $V_1$ et $V_2$. On voit que la condition nécessaire et suffisante pour que les droites $M$ et $M'$ se rencontrent est $V_1 \sin V_2 = 0$, c'est-à-dire :

$$\Sigma (\alpha_1 \sin\alpha_2 \cos\alpha_2' + \alpha_1' \sin\alpha_2' \cos\alpha_2) = 0.$$

Lorsque les droites $M$ et $M'$ sont infiniment voisines on peut poser : $\alpha' = \alpha + d\alpha$, etc. ; on a alors, en négligeant les quantités d'ordre supérieur :

$$dV^2 = \Sigma \cos^2\alpha \,.\, d\alpha^2$$

et l'on en tire :

$$\begin{cases} dV_2^2 = \Sigma \cos^2\alpha_2 \,.\, d\alpha_2^2, \\ dV_1 \,.\, dV_2 = \Sigma (\cos\alpha_2 d\alpha_1 - \alpha_1 \sin\alpha_2 d\alpha_2) \cos\alpha_2 d\alpha_2. \end{cases}$$

Si les coordonnées $\alpha_1$, $\beta_1$, $\gamma_1$ et $\alpha_2$, $\beta_2$, $\gamma_2$ sont fonctions d'un même paramètre, la droite $M$ décrira une surface réglée et la condition pour que cette surface soit développable sera exprimée par l'équation : $dV_1 dV_2 = 0$ ou :

$$\Sigma (\cos\alpha_2 d\alpha_1 - \alpha_1 \sin\alpha_2 d\alpha_2) \cos\alpha_2 d\alpha_2 = 0.$$

L'expression du codistangle de deux droites en coordonnées tripolaires, montre aussi que l'équation d'une congruence circulaire de rayon $r$, dont le pôle a pour coordonnées $a$, $b$, $c$, est:

$$\cos a \cos \alpha + \cos b \cos \beta + \cos c \cos \gamma = \cos r,$$

$\alpha$, $\beta$, $\gamma$ étant les coordonnées courantes.

3°. *Coordonnées cartésiennes.* La position d'un point sur la sphère peut aussi être rapportée à deux grands cercles fixes qui se coupent à angle droit et qui sont les axes de coordonnées (Fig. 3).

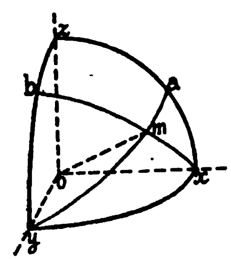



FIG. 3.

Soit $x$ le pôle du grand cercle $yz$ et $y$ celui du grand cercle $xz$; le point $z$ est l'origine des coordonnées. Etant donné un point quelconque $m$ sur la sphère, on trace les grand cercles $ym$ et $xm$ qui interceptent sur les axes de coordonnées les arcs $za$ et $zb$; les tangentes de ces arcs servent de coordonnées au point $m$. On posera donc:

$$\operatorname{tang}(za) = \xi, \qquad \operatorname{tang}(zb) = \eta.$$

Réciproquement, $\xi$ et $\eta$ étant donnés, les arcs $za$ et $zb$ sont connus et la position du point $m$ se trouve sans difficulté. Toute relation linéaire entre $\xi$ et $\eta$ représente un grand cercle de la sphère; en particulier, l'équation:

$$\frac{\xi - \xi'}{\xi'' - \xi'} = \frac{\eta - \eta'}{\eta'' - \eta'},$$

représentera le grand cercle passant par les deux points $(\xi', \eta')$ et $(\xi'', \eta'')$. Toute équation du second degré entre $\xi$ et $\eta$ représente une conique sphérique et en général, on dira qu'une courbe sphérique est de degré $m$ lorsque son équation en

$\xi$ et $\eta$ est de degré $m$; ceci revient à dire qu'une courbe sphérique est de degré $m$ lorsqu'elle est coupée en $m$ points par un grand cercle quelconque ou encore lorsqu'elle peut être considérée comme l'intersection de la sphère avec un cône concentrique de degré $m$.

De même, considérons dans l'espace trois droites fixes $X$, $Y$, $Z$ formant un trièdre trirectangle: les recticongruences $X$ et $Y$ joueront le rôle d'axes de coordonnées et se couperont suivant la droite $Z$, qui est ainsi la droite origine (Fig. 4).

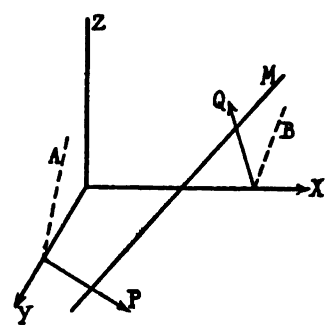



FIG. 4.

Etant donnée une droite quelconque $M$, on construira la droite $P$ qui mesure la plus courte distance de $M$ à $Y$ et par le point de rencontre de $P$ et de $Y$, on mènera une droite $A$ normale au plan $PY$; cette droite $A$ est la droite d'intersection des recticongruences $(ZX)$ et $(MY)$. On construira de même la droite $B$ intersection des congruences $(ZY)$ et $(MX)$; les coordonnées de la droite $M$ seront alors:

$$\begin{cases} \xi = \dfrac{\xi_1 + I\xi_2}{I} = \text{tang}\,(\overline{ZA}), \\ \eta = \dfrac{\eta_1 + I\eta_2}{I} = \text{tang}\,(\overline{ZB}). \end{cases}$$

Réciproquement, $\xi$ et $\eta$ étant donnés, on peut déterminer la position des droites $A$ et $B$; il suffit pour cela de trouver la valeur des codistangles:

$$(\overline{ZA}) = a = \frac{a_1 + Ia_2}{I} \quad \text{et} \quad (\overline{ZB}) = b = \frac{b_1 + Ib_2}{I}.$$

Or on a par définition : $\xi = \text{tang}\, a$, c'est-à-dire :

$$\begin{cases} \xi_1 = \dfrac{a_1}{\cos^2 a_2}, \\ \xi_2 = \text{tang}\, a_2, \end{cases} \quad \text{ou :} \quad \begin{cases} a_2 = \text{tang}^{-1} \xi_2, \\ a_1 = \dfrac{\xi_1}{1 + \xi_2^2} \end{cases}$$

et des valeurs analogues pour $b_1$ et $b_2$. Les droites $A$ et $B$ étant connues, les droites $P$ et $Q$ le sont aussi; la droite cherchée $M$ est alors la perpendiculaire commune à $P$ et à $Q$.

Toute équation linéaire: $A\xi + B\eta + C = 0$, où $A$, $B$, $C$ sont des constantes de la forme $\dfrac{p + Iq}{I}$, représentera une recticongruence dont le pôle a pour coordonnées $\dfrac{A}{C}$ et $\dfrac{B}{C}$. Soient par exemple deux droites $M'$ et $M''$ de coordonnées $(\xi', \eta')$ et $(\xi''\ \eta'')$, on demande les coordonnées de leur perpendiculaire commune $N$; la recticongruence $(M'\,M'')$ a pour équation :

$$\frac{\xi - \xi'}{\xi' - \xi''} = \frac{\eta - \eta'}{\eta' - \eta''},$$

ou:

$$(\eta' - \eta'')\,\xi - (\xi' - \xi'')\,\eta + (\xi'\eta'' - \xi''\eta') = 0$$

et son pôle qui a pour coordonnées:

$$\frac{\eta' - \eta''}{\xi'\eta'' - \xi''\eta'} \quad \text{et} \quad - \frac{\xi' - \xi''}{\xi'\eta'' - \xi''\eta'},$$

est la droite cherchée $N$.

Toute relation de la forme $f(\xi, \eta) = 0$ représente une congruence analytique dont le degré sera égal à celui de la fonction $f$ et la recticongruence tangente le long de la droite $(\xi', \eta')$ a pour équation :

$$\eta - \eta' = \frac{d\eta'}{d\xi'}(\xi - \xi').$$

*Seconde définition de la congruence analytique:*

Quel que soit le système de coordonnées employé, toute équation $\eta = F(\xi)$ entre les deux coordonnées d'une droite représente une congruence, car cette équation équivant à deux relations entre les quatre quantités $\eta_1$, $\eta_2$, $\xi_1$, $\xi_2$ dont dépend la position de la droite. Réciproquement, si l'on se donne a priori deux relations:

$$\begin{cases} \eta_1 = F_1(\xi_1, \xi_2), \\ \eta_2 = F_2(\xi_1, \xi_2), \end{cases}$$

la droite $(\xi, \eta)$ décrira une congruence, mais pour que $\eta$ soit fonction analytique de $\xi$, $\eta = F(\xi)$, il faut que:

$$\begin{cases} \eta_1 = F_1 = \xi_1 F'(\xi_2), \\ \eta_2 = F_2 = F(\xi_2), \end{cases}$$

c'est-à-dire que les fonctions $F_1$ et $F_2$ doivent satisfaire aux conditions:

$$\frac{\partial F_2}{\partial \xi_1} = 0 \quad \text{et} \quad \frac{\partial F_1}{\partial \xi_1} = \frac{\partial F_2}{\partial \xi_2}.$$

Lorsque cela a lieu on dira que la congruence est analytique. Cette nouvelle définition de la congruence analytique est un peu moins générale que la définition géométrique donnée précédemment, car il n'est pas toujours possible de trouver une congruence contenant une surface réglée donnée et dont l'équation soit analytique. En effet, soient $\xi = \frac{\xi_1 + I\xi_2}{I}$ et $\eta = \frac{\eta_1 + I\eta_2}{I}$ les deux coordonnées d'une droite $M$; supposons que les quatre quantités $\xi_1$, $\xi_2$, $\eta_1$, $\eta_2$ soient fonctions d'un même paramètre $\alpha_2$;

$$\xi_1 = \lambda(\alpha_2), \quad \xi_2 = \mu(\alpha_2), \quad \eta_1 = \nu(\alpha_2), \quad \eta_2 = \rho(\alpha_2).$$

La droite $M$ décrit alors une surface réglée $S$; cherchons l'équation de la congruence analytique qui contient $S$. Soit $\alpha_1$ un second paramètre arbitraire; si l'on pose $\frac{\alpha_1 + I\alpha_2}{I} = \alpha$ les deux équations:

$$\begin{cases} \xi = \mu(\alpha), \\ \eta = \rho(\alpha), \end{cases} \tag{9}$$

représentent une congruence analytique, car en éliminant $\alpha$ on obtiendrait une relation entre les deux coordonnées $\xi$ et $\eta$. D'autre part, en désignant les dérivées par des accents, les équations (9) sont équivalentes aux quatre relations:

$$\xi_1 = \alpha_1\mu'(\alpha_2), \quad \xi_2 = \mu(\alpha_2), \quad \eta_1 = \alpha_1\rho'(\alpha_2), \quad \eta_2 = \rho(\alpha_2),$$

qui représentent toujours une congruence, parce que les paramètres $\alpha_1$ et $\alpha_2$ sont indépendants. Mais si l'on fait dépendre $\alpha_1$ de $\alpha_2$, les équations précédentes définiront une surface réglée appartenant à la congruence (9). Or pour que cette surface réglée soit précisément la surface $S$, il faudrait que l'on eut :

$$\begin{cases} \lambda(\alpha_2) = \alpha_1 \mu'(\alpha_2), \\ \nu(\alpha_2) = \alpha_1 \rho'(\alpha_2). \end{cases} \tag{10}$$

Mais, comme on ne doit établir qu'une seule relation entre $\alpha_1$ et $\alpha_2$, le problème n'est possible que si les équations (10) se réduisent à une seule, c'est-à-dire si l'on a : $\lambda\rho' = \nu\mu'$ on ce qui est la même chose :

$$\xi_1 \, d\eta_2 - \eta_1 \, d\xi_2 = 0. \tag{11}$$

Telle est la condition que la surface $S$ doit remplir pour qu'elle fasse partie d'une congruence analytique. Lorsque cette condition est remplie, on dit que $S$ est une surface réglée analytique ; dans ce cas, on obtient l'équation de la congruence analytique dont $S$ fait partie en éliminant $\alpha$ entre les équations (9). L'une quelconque des équations (10) fait alors connaître la relation que l'on doit établir entre $\alpha_1$ et $\alpha_2$ pour que la droite $M$ décrive dans la congruence la surface donnée $S$.

L'équation (11) suppose que les dérivées $\mu'$ et $\rho'$ ne sont pas nulles, c'est-à-dire que les coordonnées $\xi_2$ et $\eta_2$ ne sont pas constantes. Si $\eta_2$ par exemple était constant, les équations de la surface $S$ seraient de la forme :

$$\xi_1 = \lambda(\alpha_2), \quad \xi_2 = \mu(\alpha_2), \quad \eta_1 = \nu(\alpha_2), \quad \eta_2 = r_2,$$

$r_2$ étant une constante. La congruence qui contient $S$ s'obtiendrait en éliminant $\alpha$ entre les équations :

$$\begin{cases} \xi = \mu(\alpha), \\ \eta = r, \end{cases}$$

$r$ étant une constante de la forme $\dfrac{r_1 + Ir_2}{I}$. On aurait donc :

$$\xi_1 = \alpha_1 \mu'(\alpha_2), \quad \xi_2 = \mu(\alpha_2), \quad \eta_1 = r_1, \quad \eta_2 = r_2,$$

c'est-à-dire que la fonction $\nu(\alpha_2)$ doit se réduire aussi à une constante $r_1$ ; l'équation de la congruence qui contient $S$ est alors simplement $\eta = r$ et la relation

$$\lambda(\alpha_2) = \alpha_1 \mu'(\alpha_2),$$

exprime toujours la loi qui relie les paramètres $\alpha_1$ et $\alpha_2$ lorsque la droite $M$ décrit dans la congruence $\eta = r$ la surface donnée $S$.

On peut remarquer que les coordonnées sphériques qui correspondent aux coordonnées $\xi$ et $\eta$ sont précisément les angles $\xi_2$ et $\eta_2$ et comme la relation entre $\xi$ et $\eta$ est la même qu'entre $\xi_2$ et $\eta_2$, on peut dire que la courbe sphérique qui correspond à une congruence analytique donnée est l'intersection d'une sphère avec le cône directeur de la congruence, dont le sommet serait au centre de la sphère.

4°. *Coordonnées intrinsèques.* Soit $r = \frac{r_1 + Ir_2}{I}$ le rayon de courbure et $s = \frac{s_1 + Is_2}{I}$ l'arc d'une surface réglée, mesuré à partir d'une certaine génératrice. Les équations intrinsèques de la surface seront de la forme:

$$r_1 = \lambda(\alpha_2), \quad r_2 = \mu(\alpha_2), \quad s_1 = \nu(\alpha_2), \quad s_2 = \rho(\alpha_2),$$

$\alpha_2$ étant un paramètre variable. Comme l'équation (11) est indépendante du système de coordonnées employé, la surface réglée sera analytique si l'on a:

$$r_1\, ds_2 - s_1\, dr_2 = 0.$$

Lorsque cette condition est remplie, la congruence analytique qui contient la surface donnée est définie par les équations: $r = \mu(\alpha)$, $s = \rho(\alpha)$. En éliminant $\alpha$, on obtient une relation entre $r$ et $s$ qui est l'équation intrinsèque de la congruence.

Si l'équation $f(\xi, \eta, p) = 0$ d'une congruence contient un paramètre arbitraire $p = \frac{p_1 + Ip_2}{I}$, cette équation définira une double famille de congruences comprenant toutes les droites de l'espace; mais si l'on se donne en outre une relation entre $p_1$ et $p_2$, l'équation $f = 0$ ne représentera plus qu'une famille simple de congruences, c'est-à-dire un complexe de droites.

## V°.—*Mécanique réglée.*

*Théorie des vectangles.* Soit $O$ le centre de la sphère fondamentale; $Ox$, $Oy$, $Oz$ trois axes rectangulaires et $\overline{OP} = V$ un segment de droite faisant avec les axes des angles $\alpha$, $\beta$, $\gamma$. Les composantes du segment $\overline{OP}$ suivant $Ox$, $Oy$ et $Oz$ sont:

$$X = V\cos\alpha, \quad Y = V\cos\beta, \quad Z = V\cos\gamma.$$

Afin de définir le segment $\overline{OP}$ et ses composantes au moyen de grandeurs situées sur la surface même de la sphère, on conviendra de représenter chaque segment issu du point $O$ au moyen d'un arc de longueur égale à celle du segment, cet arc étant porté par le grand cercle dont le plan est perpendiculaire au segment considéré.

Cet arc sera désigné sous le nom de *vecteur sphérique;* on peut, sans altérer un vecteur sphérique, le déplacer sur le grand cercle qui le porte.

D'après cette définition, étant donné un triangle trirectangle et un vecteur sphérique $V$ porté par un grand cercle quelconque (Fig. 5), on peut décomposer

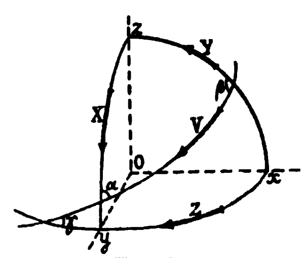


FIG. 5.

le vecteur $V$ en trois autres $X, Y, Z$ portés respectivement par les côtés du triangle trirectangle et déterminés par les formules:

$$X = V\cos\alpha, \quad Y = V\cos\beta, \quad Z = V\cos\gamma,$$

$\alpha$, $\beta$, $\gamma$ désignant les angles compris entre le vecteur $V$ et chacune de ses composantes.

Un vecteur sphérique est déterminé par deux points de la surface de la sphère; de même, dans l'espace réglé, on dira que deux droites $A$ et $B$ déterminent un *vectangle* (abréviation pour vecteur-angle): la valeur de ce vectangle sera représentée par le codistangle $(\overline{AB}) = V$. La perpendiculaire commune aux droites $A$ et $B$ est le pôle de la recticongruence $AB$; on n'altère pas un vectangle en le déplaçant dans la recticongruence qui le porte, c'est-à-dire en le faisant tourner autour ou glisser le long de son pôle, pourvu que le codistangle $(\overline{AB})$ ne change pas; c'est pourquoi l'on dira aussi que le vectangle $(\overline{AB})$ est porté par son pôle.

Etant donnés trois axes de coordonnées rectangulaires et un vectangle $V$ dans l'espace porté par une droite quelconque $P$, on pourra décomposer le vectangle $V$ en trois autres $X$, $Y$, $Z$ portés respectivement par les trois axes de coordonnées et ces trois composantes auront pour valeur:

$$X = V\cos\alpha, \quad Y = V\cos\beta, \quad Z = V\cos\gamma,$$

$\alpha$, $\beta$, $\gamma$ désignant les codistangles compris entre la droite $P$ et chacun des axes de coordonnées, puisque la figure formée par ces axes correspond sur la sphère au triangle trirectangle. En posant:

$$V = \frac{V_1 + IV_2}{I}, \quad X = \frac{X_1 + IX_2}{I}, \quad \alpha = \frac{\alpha_1 + I\alpha_2}{I}, \text{ etc. } \ldots .$$

on tire des équations précédentes:

$$\left\{\begin{array}{l} X_2 = V_2\cos\alpha_2 \\ Y_2 = V_2\cos\beta_2 \\ Z_2 = V_2\cos\gamma_2 \end{array}\right. \quad \text{et} \quad \left\{\begin{array}{l} X_1 = V_1\cos\alpha_2 - \alpha_1 V_2\sin\alpha_2, \\ Y_1 = V_1\cos\beta_2 - \beta_1 V_2\sin\beta_2, \\ Z_1 = V_1\cos\gamma_2 - \gamma_1 V_2\sin\gamma_2, \end{array}\right. \qquad (12)$$

formules qui déterminent complètement les trois vectangles composants. Réciproquement, les vectangles composants $X$, $Y$, $Z$, étant donnés, on obtiendra la valeur du vectangle résultant par l'équation complexe:

$$V = \sqrt{X^2 + Y^2 + Z^2},$$

c'est-à-dire:

$$\left\{\begin{array}{l} V_2 = \sqrt{X_2^2 + Y_2^2 + Z_2^2}, \\ V_1 V_2 = X_1X_2 + Y_1Y_2 + Z_1Z_2 \end{array}\right.$$

et la position dans l'espace de la droite $P$ qui porte ce vectangle résultant sera déterminée par les relations:

$$\cos\alpha = \frac{X}{V}, \quad \cos\beta = \frac{Y}{V}, \quad \cos\gamma = \frac{Z}{V},$$

qui déterminent $\alpha$, $\beta$, $\gamma$, c'est-à-dire les coordonnées tripolaires de $P$.

Etant donné un nombre quelconque de vectangles $V$, $V'$, $V''$, .... portés respectivement par des droites $P$, $P'$, $P''$, ...., situées d'une manière arbitraire dans l'espace, si l'on veut composer tous ces vectangles en un seul, on choisira trois axes de coordonnées rectangulaires, on décomposera chaque vectangle en trois autres ayant respectivement pour pôles les trois axes de coordonnées et l'on fera la somme algébrique des vectangles composants pour chacun des trois axes: les trois sommes $X$, $Y$, $Z$ ainsi formées seront les composantes du vectangle résultant $R$. On a donc:

$$X = \Sigma(V\cos\alpha), \quad Y = \Sigma(V\cos\beta), \quad Z = \Sigma(V\cos\gamma),$$

$\alpha$, $\beta$, $\gamma$, désignant toujours les codistangles compris entre la droite $P$ et chacun des axes de coordonnées. Les vectangles $X$, $Y$, $Z$ étant déterminés, si l'on appelle $\Pi$ le pôle du vectangle résultant $R$, et $a$, $b$, $c$ les coordonnées tripolaires de ce pôle, on aura:

$$\cos a = \frac{X}{R}, \quad \cos b = \frac{Y}{R}, \quad \cos c = \frac{Z}{R},$$

équations qui avec:

$$R^2 = X^2 + Y^2 + Z^2,$$

détermineront $R$ en grandeur et en position.

On peut aussi composer les vectangles directement, ainsi que les vecteurs sphériques, sans les rapporter à un système d'axes de coordonnées.

Considérons d'abord deux vecteurs sphériques $AC = V$ et $AC' = V'$ dont les pôles sont $P$ et $P'$ (Fig. 6). On peut toujours prendre comme origine commune

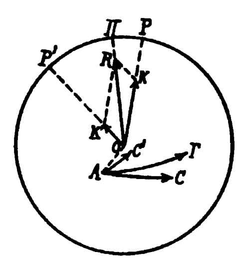



FIG. 6.

des vecteurs le pôle $A$ du grand cercle $PP'$. Le point $O$ étant le centre de la sphère, si l'on porte respectivement sur les rayons $OP$ et $OP'$ des segments $\overline{OK} = V$ et $\overline{OK'} = V'$, la résultante $R$ de ces deux segments sera portée par la diagonale $O\Pi$ du parallélogramme $KOK'$ et la grandeur ainsi que la direction de cette résultante seront déterminées par les relations:

$$\begin{cases} R^2 = V^2 + V'^2 + 2VV'\cos(\overline{PP'}), \\ V\sin(\overline{P\Pi}) = V'\sin(\overline{\Pi P'}), \end{cases} \tag{13}$$

avec $(\overline{P\Pi}) + (\overline{\Pi P'}) = (\overline{PP'})$. Or le vecteur $AC$ étant la représentation sphérique du segment $\overline{OK}$ et $AC'$ celle du segment $\overline{OK'}$, la résultante de ces vecteurs sera la représentation sphérique du segment $R$ porté par $O\Pi$; cette résultante sera donc un vecteur $A\Gamma = R$ passant par le point $A$, puisque son pôle $\Pi$ est situé sur l'arc $PP'$. La première des équations (13) détermine donc la grandeur du vecteur $A\Gamma$ et la seconde indique la position de son pôle $\Pi$ sur la sphère.

Soient maintenant $V = \frac{V_1 + IV_2}{I}$ et $V' = \frac{V_1' + IV_2'}{I}$ deux vectangles quelconques, $P$ et $P'$ leurs pôles situés arbitrairement dans l'espace. On peut, en déplaçant ces vectangles autour de leurs pôles respectifs, les amener à avoir pour origine commune la droite $A$ qui mesure la plus courte distance de $P$ à $P$ (Fig. 7). Cette droite est alors aussi la droite origine du vectangle résultant,

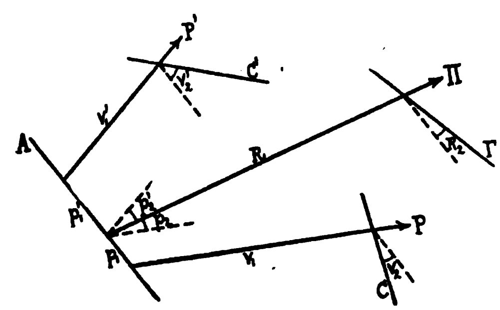


FIG. 7.

c'est-à-dire que le pôle $\Pi$ de celui-ci doit rencontrer $A$ à angle droit. Soient $C$ et $C'$ les droites extrémités des vectangles $V$ et $V'$ et $\Gamma$ celle du vectangle résultant $R$. Les mêmes équations (13) détermineront $R$ en grandeur et en position, car si l'on pose :

$$(\overline{A\Gamma}) = R = \frac{R_1+IR_2}{I}, \quad (\overline{PP'}) = \frac{a_1+Ia_2}{I}, \quad (\overline{P\Pi}) = \frac{p_1+Ip_2}{I}, \quad (\overline{\Pi P'}) = \frac{p_1'+Ip_2'}{I},$$

on aura, comme pour les vecteurs sphériques :

$$\begin{cases} \left(\frac{R_1+IR_2}{I}\right)^2 = \left(\frac{V_1+IV_2}{I}\right)^2 + \left(\frac{V_1'+IV_2'}{I}\right)^2 + 2\left(\frac{V_1+IV_2}{I}\right)\left(\frac{V_1'+IV_2'}{I}\right)\cos\left(\frac{a_1+Ia_2}{I}\right), \\ \left(\frac{V_1+IV_2}{I}\right)\sin\left(\frac{p_1+Ip_2}{I}\right) = \left(\frac{V_1'+IV_2'}{I}\right)\sin\left(\frac{p_1'+Ip_2'}{I}\right), \\ \frac{p_1+Ip_2}{I} + \frac{p_1'+Ip_2'}{I} = \frac{a_1+Ia_2}{I} \end{cases}$$

et l'on en déduira par la méthode ordinaire les six équations suivantes :

$$\begin{cases} R_2^2 = V_2^2 + V_2'^2 + 2V_2V_2'\cos a_2, \\ R_1R_2 = V_1V_2 + V_1'V_2' + (V_1V_2' + V_1'V_2)\cos a_2 - a_1V_2V_2'\sin a_2, \end{cases}$$

$$\begin{cases} V_2 \sin p_2 = V_2' \sin p_2', \\ V_1 \sin p_2 + p_1 V_2 \cos p_2 = V_1' \sin p_2' + p_1' V_2' \cos p_2', \end{cases} \tag{14}$$

avec
$$\begin{cases} p_2 + p_2' = a_2, \\ p_1 + p_1' = a_1. \end{cases}$$

Les quatre dernières équations déterminent les longueurs $p_1$, $p_1'$ et les angles $p_2$, $p_2'$, c'est-à-dire la position du pôle $\Pi$ par rapport aux pôles $P$ et $P'$, tandis que les deux premières fournissent les valeurs de $R_1$ et $R_2$, d'où dépend la position de la droite $\Gamma$.

Considérons enfin un nombre quelconque de vectangles $V$, $V'$, $V''$ .... portés respectivement par des droites $P$, $P'$, $P''$ .... données arbitrairement dans l'espace; en comparant toujours les vectangles aux vecteurs sphériques, on verra facilement que le vectangle résultant $R$ a pour valeur:

$$R^2 = \Sigma V^2 + 2\Sigma VV' \cos(\overline{PP'}).$$

*Composition des efforts ou des mouvements.* Un segment tel que $\overline{OK}$ (Fig. 6) sert à definir soit un couple, c'est-à-dire l'effort le plus général que l'on puisse exercer sur une sphère dont le centre est fixe, soit une vitesse de rotation, c'est-à-dire le mouvement le plus général que cette sphère puisse prendre autour de son centre. De même, un vectangle $(\overline{AB})$ servira à définir soit un torseur, c'est-à-dire l'effort le plus général que l'on puisse exercer sur un corps solide, soit une vitesse hélicoïdale, c'est-à-dire le mouvement le plus général que ce solide puisse prendre. On sait qu'un torseur se compose d'une force et d'un couple dont l'axe coïncide avec la direction de la force; si donc le vectangle $(\overline{AB}) = \dfrac{P + QI}{I}$ représente un torseur, la plus courte distance $P$ représentera le moment du couple et l'angle $Q$ représentera la force; cette force agit suivant l'axe du torseur, c'est-à-dire suivant le pôle du vectangle $(\overline{AB})$; toute force est ainsi définie par un angle situé dans un plan perpendiculaire à sa ligne d'action.

Si au contraire le vectangle $(\overline{AB}) = \dfrac{P + QI}{I}$ figure une vitesse hélicoïdale, la plus courte distance $P$ représentera la vitesse de translation et l'angle $Q$ la vitesse de rotation; l'axe du mouvement coïncide d'ailleurs avec le pôle du vectangle. Ce pôle reste fixe pendant le mouvement hélicoïdal, de même que le pôle d'une rotation sphérique reste immobile pendant la rotation.

Les torseurs et les vitesses hélicoïdales se composant comme les vectangles, tous les cas de composition d'efforts ou de mouvements rentreront dans le cas général de composition des vectangles. En effet, on peut regarder une force ordinaire comme un torseur dont les extrémités $A$ et $B$ se rencontrent, car si $P=0$, le torseur $(\overline{AB})=Q=$ force; si au contraire $Q$ est nul, les droites $A$ et $B$ sont parallèles et on a $(\overline{AB})=\frac{P}{I}=$ couple. De même, une vitesse de rotation est définie par un angle $Q$ et une vitesse de translation est de la forme $\frac{P}{I}$, car cette vitesse est définie par deux droites parallèles.

Soit par exemple à composer deux rotations $v_2$ et $v_2'$ dont les axes $P$ et $P'$ ne se rencontrent pas. Soit $a_1$ la plus courte distance et $a_2$ l'angle des droites $P$ et $P'$. Si l'on se reporte au cas général de composition des vectangles (Fig. 7), et que l'on pose $v_1=0$ et $v_1'=0$, la vitesse du mouvement résultant sera déterminée par les équations:

$$R_2^2=v_2^2+v_2'^2+2v_2v_2'\cos a_2,$$
$$R_1R_2=-a_1v_2v_2'\sin a_2,$$

$R_1$ étant la résultante de translation et $R_2$ celle de rotation, c'est-à-dire que la résultante est une vitesse hélicoïdale de la forme $\frac{R_1+IR_2}{I}$. La dernière équation est du reste identique à une formule de Rodrigues sur la composition des rotations; la formule de Rodrigues n'est ainsi qu'un cas particulier de la formule plus générale relative aux vectangles. Enfin, l'axe $\Pi$ du mouvement résultant, qui on l'a vu doit rencontrer $A$ à angle droit, sera déterminé en position par les équations:

$$\begin{cases} v_2\sin p_2= v_2'\sin p_2', \\ p_1v_2\cos p_2=p_1'v_2'\cos p_2', \end{cases} \quad \text{avec} \quad \begin{cases} p_2+p_2'=a_2, \\ p_1+p_1'=a_1, \end{cases}$$

qui peuvent s'écrire:

$$\begin{cases} v_2\sin p_2=v_2'\sin p_2', \\ \dfrac{p_1}{\operatorname{tang} p_2}=\dfrac{p_1'}{\operatorname{tang} p_2'}. \end{cases}$$

Or $\frac{p_2}{p_2'}$ est le rapport des angles compris entre l'axe $\Pi$ et les axes $P$ et $P'$ et $\frac{p_1}{p_1'}$ est le rapport suivant lequel $\Pi$ partage la plus courte distance de $P$ à $P'$; on reconnaît donc dans les deux dernières équations deux théorèmes bien connus sur la composition des rotations non concourantes.

Il n'est pas inutile de faire remarquer ici que les règles d'opérations relatives aux codistangles subsistent dans tous les cas particuliers, quoique les formules revêtent quelquefois une apparence paradoxale. Ainsi, la règle de multiplication:

$$\left(\frac{a_1+Ia_2}{I}\right)\left(\frac{b_1+Ib_2}{I}\right)=\frac{(a_1b_2+a_2b_1)+Ia_2b_2}{I},$$

devient pour $a_2=0$ et $b_2=0$:

$$\left(\frac{a_1}{I}\right)\left(\frac{b_1}{I}\right)=0,$$

quoique ni $a_1$ ni $b_1$ ne soient nuls. De même $\left(\frac{a_1}{I}\right)^2=0$ ne signifie pas que $a_1$ est nul, car dans la formule générale:

$$\left(\frac{a_1+Ia_2}{I}\right)^2=\frac{2a_1a_2+Ia_2^2}{I},$$

il suffit que $a_2$ s'annule pour que tout le carré s'annule aussi. Si donc $a$ représente un torseur, l'équation $a^2=0$ veut dire simplement que ce torseur se réduit à un couple de la forme $\frac{a_1}{I}$. Du reste, ces cas particuliers ne mettent aucune formule en défaut, c'est ainsi qu'on a par exemple: $\sin\left(\frac{a_1}{I}\right)=\frac{a_1}{I}$, $\cos\frac{a_1}{I}=1$ et l'on en tire encore: $\sin^2\left(\frac{a_1}{I}\right)+\cos^2\left(\frac{a_1}{I}\right)=1$, parce que $\left(\frac{a_1}{I}\right)^2=0$.

Il y a cependant un cas où les formules, quoique toujours exactes, deviennent indéterminées; c'est le cas où toutes les droites considérées dans l'espace sont parallèles entre elles; tous les codistangles sont alors de la forme $\left(\frac{a_1}{I}\right)$, mais dans ce cas il n'est plus nécessaire d'employer les méthodes du calcul réglé, car en coupant toutes les droites par un plan perpendiculaire à leur direction commune, le problème revient à une question de géométrie plane.

Nous avons dit en commençant que si les règles d'opérations relatives aux codistangles sont un peu différentes des règles ordinaires, cela tient à ce que la constitution interne de l'espace réglé n'est pas absolument identique à celle de la surface d'une sphère imaginaire; on peut ajouter maintenant que cette différence de constitution est due à l'existence dans l'espace de droites parallèles entre elles, car deux droites parallèles ont une infinité de perpendiculaires communes tandis que deux points de la sphère ne déterminent jamais qu'un seul pôle; il était donc nécessaire de faire subir aux règles d'opérations une modification qui contrebalançat les différences existant entre l'espace réglé et la sphère imaginaire, ou si l'on veut entre le symbole $I$ et le symbole $i = \sqrt{-1}$.

*Theorie des Moments.* Quoique nous n'ayons fait usage, pour la composition des vectangles, que du théorème des projections, plusieurs des formules obtenues en appliquant les lois du calcul réglé ne sont pas autre chose que l'expression du théorème des moments; telles sont les formules (12) et (14).

Il existe en effet une parenté directe entre le théorème des moments et celui des projections: considérons une sphère de centre $o$ (Fig. 8); soit $om$ un rayon

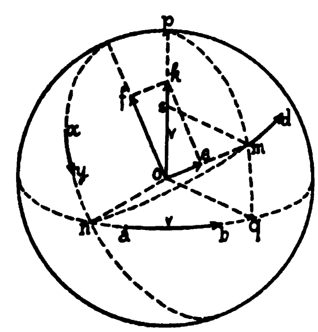


FIG. 8.

fixe et $\overline{ok} = v$ un segment porté par un rayon arbitraire $op$; soit $ab = v$ la représentation sphérique du segment $\overline{ok}$; si l'on abaisse du point $m$ un grand cercle $pmq$ perpendiculaire à $ab$, on dira que le produit $v \sin(mq)$ est le *moment sphérique* du vecteur $ab$ par rapport au point $m$ et l'on représentera ce moment par un segment $\overline{oe}$ porté sur la normale en $m$ à la sphère; or $\overline{oe} = v \sin(mq)$ $= \overline{ok} \,.\, \cos(mp)$, c'est-à-dire que $\overline{oe}$ est aussi la projection de $\overline{ok}$ sur $\overline{om}$. Donc, si l'on considère plusieurs vecteurs sphériques $ab$, $a'b'$, $a''b''$, etc., situés arbitrairement sur la sphère, et un point fixe $m$, on peut dire que le moment sphérique du vecteur résultant $AB$ sera égal à la somme algébrique: $\overline{OE} = \overline{oe} + \overline{oe'} + \overline{oe''} +$ etc. des moments sphériques des vecteurs composants; car $\overline{oe}$, $\overline{oe'}$, $\overline{oe''}$, etc. étant les projections sur $om$ des segments $\overline{ok}$, $\overline{ok'}$, $\overline{ok''}$, etc. qui correspondent aux vecteurs $ab$, $a'b'$, $a''b''$, etc., leur somme $\overline{OE}$ sera la projection du segment résultant $\overline{OK}$ qui correspond au vecteur résultant $AB$. Si maintenant le rayon de la sphère croît indéfiniment, la surface de celle-ci deviendra un plan, le vecteur $ab$ deviendra un vecteur rectiligne et le sinus de $(mq)$ deviendra égal à $mq$, c'est-à-dire que le moment sphérique deviendra le moment ordinaire par rapport à un point. Il y a donc identité entre le théorème des projections sur une droite et le théorème des moments dans un plan, considéré comme la limite vers laquelle tend le théorème des moments sphériques lorsque le rayon de la sphère croît indéfiniment. Pour pouvoir traduire dans l'espace réglé le théorème des moments sphériques, il faut que toutes les grandeurs que l'on considère soient situées sur la surface même de la sphère. On remplacera donc le segment $\overline{oe}$ par sa représentation sphérique, c'est-à-dire que l'on figurera le moment sphérique d'un vecteur $ab$ par rapport à un point $m$ au moyen d'un vecteur $xy = \overline{oe} = ab . \cos(mp)$, ce vecteur étant porté par le grand cercle dont le pôle est le point $m$. Si maintenant $(\overline{AB}) = v = \frac{v_1 + Iv_2}{I}$ est un vectangle arbitraire dans l'espace porté par une droite $P$ et si $M$ est une autre droite fixe formant avec $P$ un codistangle $(\overline{MP}) = p = \frac{p_1 + Ip_2}{I}$, on dira que le produit: $(\overline{AB}) \cos(\overline{MP})$ est le *moment* du vectangle $(\overline{AB})$ par rapport à la droite $M$, et l'on représentera ce moment par un vectangle $(\overline{XY}) = m = \frac{m_1 + Im_2}{I}$ porté par la droite $M$. Le théorème des moments dans l'espace réglé s'énoncera donc comme suit: Etant donnés plusieurs vectangles $v$, $v'$, $v''$, etc. portés par des droites $P$, $P'$, $P''$, etc. situées d'une manière arbitraire dans l'espace, si l'on prend les moments $m$, $m'$, $m''$, etc. de ces vectangles par rapport à une droite $M$, tous ces moments seront portés par la droite $M$ et leur somme algébrique sera le moment de la résultante $V$ des vectangles $v$. On aura donc:

$$\left(\frac{V_1 + IV_2}{I}\right) \cos\left(\frac{P_1 + IP_2}{I}\right) = \Sigma \left(\frac{v_1 + Iv_2}{I}\right) \cos\left(\frac{p_1 + Ip_2}{I}\right),$$

c'est-à-dire:

$$\begin{cases} V_2 \cos P_2 = \Sigma v_2 \cos p_2, \\ V_1 \cos P_2 - P_1 V_2 \sin P_2 = \Sigma (v_1 \cos p_2 - p_1 v_2 \sin p_2). \end{cases}$$

Si l'on considère $v_1$ comme le moment d'un couple porté par $P$ et $v_2$ comme une force agissant suivant $P$, la première de ces formules n'est que le théorème des projections sur une droite $M$ et la seconde, le théorème des moments par rapport à une droite $M$. Du reste nous avions déjà rencontré ces formules à propos de la composition des vectangles; en effet il n'y a pas de différence entre le moment d'un vectangle $V$ par rapport à une droite $M$ et la projection de $V$ sur $M$ puisque l'un et l'autre sont égaux à $V \cos(\overline{MP})$.

Mais il existe un second théorème des projections, obtenu en projetant un contour fermé sur un plan et non pas sur une droite; et ce second théorème conduira aussi à un second théorème sur les moments. Revenons en effet à la figure (8) et projetons le segment $\overline{ok}$ sur un plan perpendiculaire au rayon fixe $om$; le segment $\overline{of}$ qui représente cette projection est situé dans le plan $mop$ et a pour valeur: $\overline{of} = \overline{ok} \sin(mp) = \overline{ok} \,.\, \overline{ms}$, en appelant $\overline{ms}$ la distance du point $m$ au segment $\overline{ok}$; on voit donc que $\overline{of}$ est égal au moment du segment $\overline{ok}$ par rapport au point $m$. Si l'on considère plusieurs segments $\overline{ok}$, $\overline{ok'}$, $\overline{ok''}$, etc. issus du point $o$, la projection $\overline{oF}$ de leur résultante $\overline{oK}$ sur un plan perpendiculaire à $om$ est la résultante des projections $\overline{of}$, $\overline{of'}$, $\overline{of''}$, etc. des segments donnés. La représentation sphérique du segment $of$ est un vecteur $\overline{md} = \overline{of}$ porté par le grand cercle passant par le point $m$ et par le pôle $n$ du grand cercle $mp$; le vecteur sphérique $md$ représente donc le moment du segment $\overline{ok}$ par rapport au point $m$, non seulement en grandeur, mais aussi en position, puisque ce vecteur est normal en $m$ au plan $mok$. Donc le second théorème des projections est identique au théorème des moments par rapport à un point $m$ de plusieurs segments concourants $\overline{ok}$, $\overline{ok'}$, $\overline{ok''}$, etc.

Pour ne conserver que des grandeurs situées sur la surface de la sphère, on remplacera le segment $\overline{ok}$ par sa représentation sphérique $ab$ et l'on dira que le vecteur $md = \overline{ok} \,.\, \sin mp = \overline{ab} \,.\, \cos(mq)$ est le *co-moment* du vecteur $ab$ par rapport au point $m$, afin de distinguer ce moment du moment sphérique. Si l'on prend les co-moments de plusieurs vecteurs $ab$, $a'b'$, $a''b''$, etc. par rapport à un même point $m$, on obtient des vecteurs $md$, $md'$, $md''$, etc. issus du point $m$ et dont la résultante $mD$ est le co-moment du vecteur résultant $AB$.

De même, étant donnés une droite fixe $M$ et un vectangle $(\overline{AB})$ porté par une droite $P$, on dira que le produit $(\overline{AB}) \sin(\overline{MP})$ est le co-moment du vectangle $(\overline{AB})$ par rapport à la droite $M$, et l'on représentera ce moment par un vectangle $(\overline{MD})$ porté par la perpendiculaire commune à $M$ et à la droite $N$ qui mesure la plus courte distance de $M$ à $P$. Si l'on construit les co-moments de plusieurs vectangles par rapport à une même droite $M$, tous ces moments auront pour origine la droite $M$ et leur résultante sera le co-moment du vectangle résultant; tel est le second théorème des moments dans l'espace réglé.

Pour que le co-moment d'un vectangle $(\overline{AB})$ s'annule, il faut que $(\overline{AB}) = 0$ ou bien $(\overline{MP}) = 0$, c'est-à-dire que la droite $M$ coïncide avec le pôle $P$ du vectangle $(\overline{AB})$. Ainsi, si l'on a un système de vectangles dont la résultante n'est pas nulle, et que la résultante des co-moments de ces vectangles par rapport a une certaine droite $M$ soit nulle, la droite $M$ ne peut être que le pôle du vectangle équivalent au système donné, c'est-à-dire l'axe central du système. L'axe central est ainsi la seule droite par rapport à laquelle le co-moment du système est nul.

*Equilibre d'un corps solide.* La théorie de l'équilibre n'offre aucune difficulté, car si $V$, $V'$, $V''$, etc. sont des torseurs portés respectivement par des droites $P$, $P'$, $P''$, etc. et appliqués à un même corps rigide, il suffit pour que celui-ci soit en équilibre que le torseur résultant $R$ soit nul. Si donc $X$, $Y$, $Z$ son trois axes de coordonnées rectangulaires, on devra avoir:

$$\begin{cases} \Sigma V \cos(\overline{PX}) = 0, \\ \Sigma V \cos(\overline{PY}) = 0, \\ \Sigma V \cos(\overline{PZ}) = 0 \end{cases}$$

et ces trois équations complexes fourniront les six équations de l'équilibre.

*Mouvement d'une droite.* La position d'une droite $M$ dans l'espace est définie par deux coordonnées complexes $\xi = \frac{\xi_1 + I\xi_2}{I}$ et $\eta = \frac{\eta_1 + I\eta_2}{I}$; si l'on exprime ces coordonnées en fonction d'une troisième variable complexe $t = \frac{t_1 + It_2}{I}$ en posant: $\xi = \phi(t)$ et $\eta = \psi(t)$, la trajectoire de la droite $M$ sera une congruence analytique; on peut considérer la variable indépendante $t$ comme un *temps complexe.* Le temps complexe peut être comparé à un temps à deux dimensions, et en effet si le temps avait deux dimensions, la trajectoire d'un point en mouvement serait une surface et non une ligne, et la trajectoire d'une droite serait une congruence; avec un temps complexe cette congruence trajectoire est en outre analytique.

Si l'on établit une relation entre $t_1$ et $t_2$, le temps redevient simple ou si l'on veut, le mouvement n'a plus qu'un seul degré de liberté, la trajectoire de la droite $M$ est alors une surface réglée, faisant partie de la congruence trajectoire.

Réciproquement, tout mouvement à un degré de liberté d'une droite $M$ est défini si l'on connaît les quatre coordonnées $\xi_1$, $\xi_2$, $\eta_1$, $\eta_2$ en fonction d'un temps simple $t_2$:

$$\xi_1 = \lambda(t_2), \quad \xi_2 = \mu(t_2), \quad \eta_1 = \nu(t_2), \quad \eta_2 = \rho(t_2).$$

Mais on a vu que ce mouvement n'est analytique, c'est-à-dire ne fait partie d'un mouvement analytique à deux degrés de liberté, que dans le cas où: $\xi_1 d\eta_2$

$-\eta_1 d\xi_2 = 0$. Dans ce cas, le mouvement à deux degrés de liberté dont fait partie le mouvement de $M$ est défini par les équations:

$$\begin{cases} \xi = \mu(t), \\ \eta = \rho(t). \end{cases}$$

On peut donc considérer le mouvement à un degré de liberté de la droite $M$ comme défini par ces deux dernières équations, pourvu que l'on établisse en outre entre $t_1$ et $t_2$ la relation: $t_1\mu'(t_2) = \lambda(t_2)$ ou la relation équivalente: $t_1\rho'(t_2) = \nu(t_2)$.

Le temps complexe peut être représenté par une recticongruence décrite par une droite mobile $A$ à partir d'une certaine origine $o$. Chaque position de la droite $A$ définit une *époque* et la durée $t$ comprise entre l'époque $o$ et l'époque $A$ est égale au codistangle $(\overline{OA}) = \frac{t_1 + It_2}{I}$. Considérons une droite qui décrit une congruence analytique; soient $M_0$ et $M$ les positions de cette droite aux époques $o$ et $t$, l'arc $M_0M$ de la congruence trajectoire est évidemment fonction du temps complexe; si l'on désigne cet arc par $s$, on a donc: $s = f(t)$. On sait du reste que l'arc $s = \frac{s_1 + Is_2}{I}$ ne dépend pas de la surface réglée qui mène de $M_0$ à $M$. Si l'on a:

$$s = a + bt,$$

$a$ et $b$ étant des constantes complexes, on dit que le mouvement de $M$ sur sa trajectoire est *uniforme*. Le mouvement sera donc uniforme si l'on a:

$$\frac{s_1 + Is_2}{I} = \frac{a_1 + Ia_2}{I} + \frac{b_1 + Ib_2}{I} \frac{t_1 + It_2}{I},$$

c'est-à-dire:

$$\begin{cases} s_2 = a_2 + b_2 t_2, \\ s_1 = a_1 + b_1 t_2 + b_2 t_1. \end{cases}$$

Ces conditions sont d'ailleurs indépendantes de la forme de la congruence trajectoire. Si l'on pose $t_1 = F(t_2)$ la droite $M$ décrit une certaine surface réglée dans la congruence, et ce mouvement à un degré de liberté est dit uniforme si l'arc $s = \frac{s_1 + Is_2}{I}$ décrit pendant le temps $t_2$ satisfait aux conditions:

$$\begin{cases} s_2 = a_2 + b_2 t_2, \\ s_1 = a_1 + b_1 t_2 + b_2 F(t_2). \end{cases}$$

On dit que le mouvement de la droite $M$ est *rectiligne* lorsque la trajectoire de cette droite est une recticongruence. S'il s'agit d'un mouvement à un degré de liberté, il sera dit rectiligne lorsque la droite $M$ décrit un conoïde droit.

Enfin le mouvement de $M$ est rectiligne et uniforme lorsque la trajectoire est une recticongruence et que les arcs $s$ croissent proportionnellement aux temps $t$. Si le mouvement n'a qu'un degré de liberté, on dit qu'il est rectiligne et uniforme toutes les fois que la droite $M$ décrit un conoïde droit en tournant uniformément autour de l'axe du conoïde. En effet, les équations du mouvement sont alors de la forme:

$$\begin{cases} s_2 = a_2 + b_2 t_2, \\ s_1 = \phi(t_2), \end{cases}$$

$\phi$ étant une fonction quelconque, et en posant:

$$t_1 = \frac{1}{b_2}\left[\phi(t_2) - a_1 - b_1 t_2\right],$$

on aura bien:

$$\begin{cases} s_2 = a_2 + b_2 t_2, \\ s_1 = a_1 + b_1 t_2 + b_2 t_1, \end{cases}$$

c'est-à-dire que le mouvement considéré fait bien partie d'un mouvement rectiligne et uniforme $s = a + bt$. On verrait de même qui si la droite $M$ décrit une surface réglée analytique quelconque, il suffit que $s_2$ soit proportionnel au temps $t_2$ pour que le mouvement soit uniforme.

Si $M$ et $M'$ sont les positions d'une droite sur sa congruence trajectoire aux époques $t$ et $t + dt$ et si le codistangle $(\overline{MM'}) = ds$, le rapport $v = \frac{ds}{dt}$ sera la *vitesse* de la droite au temps $t$; cette vitesse sera représentée par un vectangle $(\overline{MA})$ porté à partir de la droite $M$ par la tangente à la congruence trajectoire; le pôle $T$ du vectangle $(\overline{MA})$ est donc la perpendiculaire commune aux génératrices $M$ et $M'$ et ce vectangle représente une *vitesse de torsion*,* car tout déplacement infinitésimal de $M$ dans sa congruence trajectoire est une torsion infiniment petite autour de $T$.

De la formule: $v = \frac{ds}{dt}$, ou: $\frac{v_1 + Iv_2}{I} = \frac{ds_1 + Ids_2}{dt_1 + Idt_2}$, on tire:

$$\begin{cases} v_2 = \dfrac{ds_2}{dt_2}, \\ v_1 = \dfrac{ds_1\,dt_2 - ds_2\,dt_1}{dt_2^2}. \end{cases}$$

*Il n'y a pas de différence entre une vitesse de torsion et une vitesse hélicoïdale, si ce n'est que la première se rapporte à un mouvement à deux degrés de liberté et la seconde à un mouvement à un degré.

Ainsi, dans le vectangle $(\overline{MA})$ qui figure la vitesse $v$, l'angle $v_2$ compris entre $M$ et $A$ représente la vitesse de rotation de la droite $M$ autour de la droite $T$, mais la distance $v_1$ comprise entre $M$ et $A$ ne représente pas la vitesse de translation de la droite $M$, à moins que l'on ait $dt_1 = 0$ ou $t_1 = \text{const.}$; alors $v_1 = \frac{ds_1}{dt_2}$, c'est-à-dire que $v_1$ est la vitesse de translation de $M$ le long de $T$, lorsque $M$ décrit la surface réglée déterminée par la condition $t_1 = const.$

Lorsque les équations du mouvement d'une droite $M$ sont données dans un certain système de coordonnées, on peut calculer les composantes de la vitesse de $M$ suivant chacune des deux coordonnées. Soient par exemple:

$$\rho = \phi(t) \text{ et } \omega = \psi(t),$$

les équations du mouvement en coordonnées polaires, la vitesse est:

$$v = \frac{ds}{dt} = \frac{\sqrt{d\rho^2 + \sin^2 \rho \,.\, d\omega^2}}{dt},$$

ses composantes sont donc: $\frac{d\rho}{dt}$ suivant le méridien et $\sin \rho \frac{d\omega}{dt}$ suivant le parallèle.

Pour définir l'*accélération* d'une droite, considérons d'abord un point $m$ mobile sur la sphère (Fig. 9). Soient $mv$ et $mv'$ les arcs représentant sa vitesse

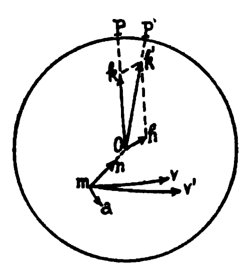



FIG. 9.

aux temps $t$ et $t + dt$; ces arcs sont les représentations sphériques des segments $\overline{ok} = v$, $\overline{ok'} = v'$, qui figurent la rotation du point $m$ autour des pôles $P$ et $P'$ des arcs $mv$ et $mv'$. Soit $\overline{oh}$ un segment tel que $\overline{oh}$ et $\overline{ok}$ aient pour résultante $\overline{ok'}$. Comme $\overline{oh}$ est situé dans le plan $kok'$, sa représentation sphérique $ma$ passe par $m$; le vecteur $ma$, qui composé avec $mv$ donnerait la vitesse $mv'$ est l'*accélération sphérique* élémentaire du point $m$. L'accélération sphérique n'est pas autre chose que l'accélération ordinaire, lorsqu'on fait abstraction de la composante $\overline{mn} = v^2 . \overline{mo}$, normale à la sphère; il n'y a pas lieu en effet de considérer cette composante, car nous ne nous occupons du mouvement du point $m$ qu'autant qu'il a lieu sur la surface de la sphère et sans nous préoccuper des réactions normales à cette surface, qui du reste sont détruites par la fixité du centre. On peut décomposer l'accélération sphérique du point $m$ en deux vecteurs $\phi$ et $\psi$ portés l'un par le grand cercle $mv$ tangent à la trajectoire, l'autre par le grand cercle normal en $m$ à cette trajectoire; ces deux composantes seront égales aux composantes de $\overline{oh}$ suivant $\overline{ok}$ et suivant une perpendiculaire à $\overline{ok}$. Si donc $ds$ est l'élément d'arc décrit par $m$ pendant le temps $dt$ et $d\omega$ l'angle de contingence de la trajectoire, c'est-à-dire l'angle $vmv'$ ou $kok'$, les composantes de $ma$ seront: $dv$ suivant la tangente et $vd\omega$ suivant la normale. Or $vd\omega = v\dfrac{d\omega}{ds}ds = v^2\dfrac{d\omega}{ds}dt$ d'autre part $\dfrac{d\omega}{ds} = \dfrac{1}{\operatorname{tang}\rho}$, $\rho$ étant le rayon de courbure sphérique de la trajectoire; on aura par suite, pour l'accélération sphérique tangente, $\phi = \dfrac{dv}{dt}$ et pour l'accélération sphérique normale, $\psi = \dfrac{v^2}{\operatorname{tang}\rho}$.

Il en résulte que si $M$ est une droite mobile qui décrit une congruence analytique, $(\overline{MV})$ et $(\overline{MV'})$ les vectangles qui figurent la vitesse de cette droite aux époques $t$ et $t+dt$, l'*accélération* élémentaire de la droite sera représentée par un vectangle $(\overline{MA})$, tel que $(\overline{MA})$ et $(\overline{MV})$ aient pour résultante $(\overline{MV'})$; et si $\rho$ est le rayon de courbure de la congruence trajectoire, les composantes de cette accélération suivant la tangente et suivant la normale seront des vectangles: $\phi = \dfrac{dv}{dt}$ et $\psi = \dfrac{v^2}{\operatorname{tang}\rho}$, c'est-à-dire, avec les notations ordinaires:

$$\frac{\phi_1 + I\phi_2}{I} = \frac{dv_1 + Idv_2}{dt_1 + Idt_2} \quad \text{et} \quad \frac{\psi_1 + I\psi_2}{I} = \frac{\left(\dfrac{v_1 + Iv_2}{I}\right)^2}{\operatorname{tang}\left(\dfrac{\rho_1 + I\rho_2}{I}\right)},$$

relations d'où l'on tire après réduction:

$$\left\{\begin{array}{l} \phi_2 = \dfrac{dv_2}{dt_2}, \\ \phi_1 = \dfrac{dv_1\,dt_2 - dv_2\,dt_1}{dt_2^2} \end{array}\right. \quad \text{et} \quad \left\{\begin{array}{l} \psi_2 = \dfrac{v_2^2}{\operatorname{tang}\rho_2}, \\ \psi_1 = (v_1 \sin 2\rho_2 - v_2\rho_1)\dfrac{v_2}{\sin^2\rho_2}. \end{array}\right.$$

Lorsque l'accélération tangentielle est nulle, la vitesse est constante et le mouvement est uniforme. Lorsque l'accélération normale s'annule, on a $\operatorname{tang}\rho = \infty$, c'est-à-dire $\rho = \frac{\pi}{2}$ ; dans ce cas la trajectoire présente une *droite d'inflexion*, c'est-à-dire que la recticongruence tangente est osculatrice. Si l'accélération normale est constamment nulle, la trajectoire est une recticongruence et le mouvement est rectiligne.

Considérons encore le mouvement circulaire uniforme: Soit $c$ le pôle d'un petit cercle que décrit un point $m$ sur la sphère avec une vitesse sphérique constante $v$, on a: $\phi = 0$ et $\psi = \frac{v^2}{\operatorname{tang} r}$, mais si l'on désigne par $\omega$ la vitesse angulaire du point $m$ autour du pôle $c$, on a aussi: $v = \omega . \sin r$, d'où:

$$\psi = \tfrac{1}{2}\omega^2 \sin 2r.$$

Cette même formule donnera donc aussi la valeur de l'accélération d'une droite mobile $M$ qui décrit une congruence circulaire de rayon $r = \frac{r_1 + Ir_2}{I}$ avec une *vitesse angulaire* $\omega = \frac{\omega_1 + I\omega_2}{I}$ autour du pôle $c$ de la congruence; cette vitesse angulaire est le codistangle décrit pendant l'unité de temps par le *rayon* aboutissant à la droite $M$, c'est-à-dire par la normale au cylindre focal au point où $M$ touche ce cylindre. L'accélération $\psi = \frac{\psi_1 + I\psi_2}{I}$ de la droite $M$ est donc déterminée par les relations:

$$\begin{cases} \psi_2 = \tfrac{1}{2}\omega_2^2 \sin 2r_2, \\ \psi_1 = \omega_2(\omega_1 \sin 2r_2 + r_1\omega_2 \cos 2r_2). \end{cases}$$

*Mouvement d'un corps solide libre.* Nous considérons un corps solide comme formé d'un nombre fini ou infini de droites rigides invariablement reliées les unes aux autres. Pour se rendre compte de la distribution des vitesses dans un corps solide en mouvement, il suffit de se rappeler que toute droite de l'espace décrit à chaque instant un élément de congruence circulaire autour de l'axe central $X$ avec la même vitesse angulaire $\omega$; si donc $M$ est une droite quelconque, faisant avec l'axe central un codistangle $(\overline{MX}) = r$, la vitesse de cette droite sera $v = \omega . \sin r$ et le vectangle qui représente cette vitesse sera porté à partir de $M$ par le pôle $T$ de la tangente à la congruence circulaire; comme le pôle de la normale à cette congruence est la droite $N$ qui mesure la plus courte distance entre $M$ et $X$, la droite $T$ est la droite qui forme avec $M$ et $N$ un trièdre trirectangle.

Pour se rendre compte de la distribution des accélérations, il faut étudier d'abord les accélérations des différents points de la surface d'une sphère mobile autour de son centre. Soit $\omega$ la vitesse angulaire de la sphère autour de l'axe instantané $\overline{ox}$ (Fig. 10), $\alpha$ son accélération angulaire portée par le rayon $\overline{oa}$ et $\lambda$

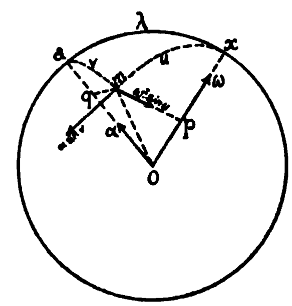



FIG. 10.

l'angle $xoa$; le plan $xoa$ est alors le plan tangent commun aux deux cônes qui en roulant l'un sur l'autre produisent le mouvement de la sphère. On peut définir la position de tout point $m$ situé sur la sphère par ses distances $mx = u$ et $ma = v$ aux points fixes $x$ et $a$.

Si l'on abaisse du point $m$ les perpendiculaires $\overline{mp}$ et $\overline{mq}$ sur les rayons $\overline{ox}$ et $\overline{oa}$, l'accélération du point $m$, telle qu'elle est exprimée par le théorème de Rivals, est la résultante de deux segments: l'un égal à $\omega^2 . \overline{mp}$ ou $\omega^2 . \sin u$, porté par $\overline{mp}$, l'autre égal à $\alpha . \overline{mq}$ ou $\alpha . \sin v$, perpendiculaire au plan $moa$.

Pour que l'accélération sphérique d'un point soit nulle, il faut que l'accélération totale de ce point soit normale à la sphère. Pour cela, il faut d'abord que le rayon $\overline{mo}$ soit dans le même plan que les deux segments qui composent l'accélération, c'est-à-dire que l'angle $amx$ soit droit; le triangle $amx$ étant alors rectangle en $m$, on aura:

$$\cos u \cos v = \cos \lambda .$$

Telle est l'équation du lieu des points dont l'accélération sphérique tangentielle est nulle; ce lieu est une conique sphérique ayant pour sommets les points $a$ et $x$. Comme l'angle $pmo$ est le complément de $u$, la résultante des segments $\omega^2 . \sin u$ et $\alpha . \sin v$ coïncidera avec $\overline{mo}$, si l'on a en outre:

$$\omega^2 \sin u \cos u = \alpha \sin v .$$

Cette équation jointe à la précédente détermine les points $(u, v)$ dont l'accélération sphérique est nulle, c'est-à-dire les centres d'accélération; ces centres sont au nombre de trois. On verrait facilement que les points qui n'ont pas d'accélération sphérique normale, c'est-à-dire les points dont la trajectoire présente un point d'inflexion sphérique, sont sur un cône du troisième degré.

De même dans l'espace réglé, étant donnés deux vectangles $\omega$ et $\alpha$ dont l'un représente la vitesse angulaire et l'autre l'accélération angulaire du système mobile à une époque déterminée, si les pôles $X$ et $A$ de ces vectangles forment entre eux un codistangle $(\overline{XA}) = \lambda$, on démontrera comme pour la sphère que le lieu des droites dont l'accélération tangentielle est nulle est une congruence elliptique admettant les droites $X$ et $A$ pour sommets; que le lieu des droites qui sont d'inflexion sur leur trajectoire est une congruence analytique du troisième ordre, et enfin qu'il existe trois droites ne possédant aucune accélération; la position de ces droites sera déterminée par les mêmes équations que précédemment, dans lesquelles $u$ et $v$ désigneront les codistangles compris entre l'une de ces droites et les pôles $X$ et $A$.

La considération des accélérations dans le mouvement sphérique conduit à une formule analogue à celle de Savary pour le mouvement plan: Soient $r$ et $\theta$ les coordonnées polaires sphériques d'un point $m$, en prenant pour origine le pôle instantané de rotation $x$ et en comptant les angles $\theta$ à partir du grand cercle tangent aux deux courbes sphériques $c$ et $c'$ qui pendant le mouvement roulent l'une sur l'autre sans glisser; soient $R$ et $R'$ les rayons de courbure sphériques des courbes $c$ et $c'$ au point $x$ et $\rho$ le rayon de courbure de la trajectoire du point $m$, on sait que l'on a:

$$\frac{1}{\text{tang}\,(\rho - r)} + \frac{1}{\text{tang}\,r} = \frac{1}{\sin\theta}\left(\frac{1}{\text{tang}\,R} + \frac{1}{\text{tang}\,R'}\right).$$

Cette formule permettra donc aussi de déterminer l'axe de courbure $K$ de la surface engendrée par une droite $M$ de l'espace, lorsque le mouvement est défini au moyen des surfaces réglées $c$ et $c'$ qui en virant l'une sur l'autre produisent le mouvement du corps solide. En effet, soit $X$ la génératrice de contact de $c$ et $c'$, $T$ le pôle de la tangente commune à $c$ et $c'$; la normale à la surface réglée décrite par $M$ a pour pôle une droite $N$ qui rencontre $M$ et $X$ à angle droit et si l'on désigne par $S$ et $S'$ les axes de courbure de $c$ et $c'$ relatifs à la génératrice $X$, on aura:

$$r = (\overline{MX}), \quad \theta = (\overline{NT}), \quad R = (\overline{XS}), \quad R' = (\overline{XS'})$$

et la formule donnée plus haut fournira la valeur de $\rho$, c'est-à-dire du codistangle $(\overline{MK})$ qui porté sur la normale $N$ à partir de $M$ détermine la position de l'axe de courbure $K$.*

Pour faire l'étude analytique du mouvement d'un système invariable libre, on considère un trièdre trirectangle $X$, $Y$, $Z$, fixe dans le système et un trièdre trirectangle $X'$, $Y'$, $Z'$, fixe dans l'espace (Fig. 11). On peut définir la position du

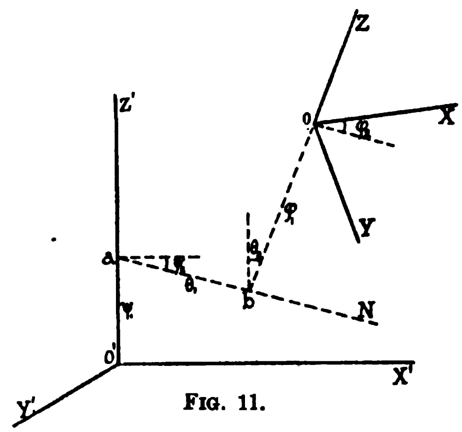



FIG. 11.

trièdre mobile par rapport au trièdre fixe au moyen de trois codistangles, qui sont dans l'espace réglé ce que les angles d'Euler sont sur la sphère : soit en effet $N$ la perpendiculaire commune aux axes $Z$ et $Z'$, $a$ et $b$ les pieds de cette perpendiculaire sur $Z'$ et sur $Z$; si l'on pose: $\overline{o'a} = \psi_1$, $\overline{ab} = \theta_1$, $\overline{bo} = \phi_1$; angle $(NX') = \psi_2$, angle $(ZZ') = \theta_2$, angle $(NX) = \phi_2$, les trois codistangles:

$$\psi = \frac{\psi_1 + I\psi_2}{I}, \quad \theta = \frac{\theta_1 + I\theta_2}{I}, \quad \phi = \frac{\phi_1 + I\phi_2}{I},$$

détermineront complètement la position du trièdre $o$ par rapport au trièdre $o'$, car on peut amener $O'X'Y'Z'$ en coïncidence avec $OXYZ$ au moyen de trois torsions $\psi$, $\theta$, $\phi$, ayant respectivement pour pôles les droites $Z'$, $N$ et $Z$.

La torsion instantanée du système invariable en mouvement peut se décomposer à chaque instant en trois torsions:

$$p = \frac{p_1 + Ip_2}{I}, \quad q = \frac{q_1 + Iq_2}{I}, \quad r = \frac{r_1 + Ir_2}{I},$$

ayant respectivement pour pôles les droites $X$, $Y$, $Z$; si donc on connaît la tor-

* Voir, pour la construction géométrique de cet axe de courbure, vol. XVIII, p. 321.

sion instantanée $p$, $q$, $r$, en fonction du temps (complexe), la position du système invariable sera déterminée en fonction du temps par les équations usuelles du mouvement sur une sphère :

$$\left\{\begin{aligned} p &= \sin\phi\ \sin\theta\frac{d\psi}{dt} + \cos\phi\frac{d\theta}{dt}, \\ q &= \cos\phi\ \sin\theta\frac{d\psi}{dt} - \sin\phi\frac{d\theta}{dt}, \\ r &= \qquad \cos\theta\frac{d\psi}{dt} + \qquad \frac{d\phi}{dt}. \end{aligned}\right. \tag{15}$$

Ces équations définissent un mouvement à deux degrés de liberté, puisque le temps $t$ est de la forme $\frac{t_1 + It_2}{I}$.

Si l'on se donne les trois torsions instantanées $p$, $q$, $r$ en fonction d'un temps simple $t_2$, le mouvement du corps n'aura plus qu'un seul degré de liberté, mais l'on peut toujours se servir des équations précédentes en considérant le mouvement à deux degrés de liberté dont fait partie le mouvement donné. Ainsi si l'on pose :

$$p_1 = \lambda(t_2), \quad q_1 = \mu(t_2), \quad r_1 = \nu(t_2),$$
$$p_2 = \rho(t_2), \quad q_2 = \sigma(t_2), \quad r_2 = \tau(t_2),$$

on peut supposer que le temps ordinaire $t_2$ fait partie d'un temps complexe $t = \frac{t_1 + It_2}{I}$, et l'on aura :

$$p = \rho(t), \quad q = \sigma(t), \quad r = \tau(t),$$

à condition que :

$$\lambda(t_2) = t_1\rho'(t_2), \quad \mu(t_2) = t_1\sigma'(t_2), \quad \nu(t_2) = t_1\tau'(t_2),$$

c'est-à-dire :

$$\frac{dp_2}{p_1} = \frac{dq_2}{q_1} = \frac{dr_2}{r_1}.$$

Lorsque ces deux conditions sont remplies, on dit que le mouvement est analytique : on a vu en effet qu'il faut une condition pour que la trajectoire d'une droite soit une surface réglée analytique ; or si deux droites décrivent chacune une surface analytique, il en est de même de toute autre droite, parce que deux droites déterminent complètement la position d'un corps solide.

On posera donc : $p = \rho(t)$, $q = \sigma(t)$, $r = \tau(t)$, dans les équations (15) et en intégrant celles-ci on obtiendra $\psi$, $\theta$, $\phi$ en fonction de $t$ ; mais comme on a en outre entre $t_1$ et $t_2$ la relation: $\lambda(t_2) = t_1 \rho'(t_2)$ (ou l'une des deux autres relations équivalentes), les trois codistangles $\psi$, $\theta$, $\phi$ ne dépendent plus que du temps $t_2$; on connaît ainsi la position du corps pour chaque valeur de $t_2$.

Enfin, pour avoir la distribution des vitesses dans le corps en mouvement, on définira la position d'une droite quelconque du corps par ses coordonnées tripolaires $\alpha$, $\beta$, $\gamma$ relatives aux axes $X$, $Y$, $Z$; les composantes de la vitesse de cette droite suivant ces axes seront alors données par les formules complexes:

$$\begin{cases} V_x = q\cos\gamma - r\cos\beta, \\ V_y = r\cos\alpha - p\cos\gamma, \\ V_z = p\cos\beta - q\cos\alpha. \end{cases}$$

Ces exemples sont suffisants pour indiquer les principales applications du calcul réglé à la statique et à la cinématique; dans un prochain article nous traiterons de la dynamique des corps solides sous le même point de vue.

# *Note on Mr. A. B. Basset's Paper, "Theory of the Action of Magnetism on Light."**

By Joseph Larmor.

I have recently been favored by Mr. Basset with a copy of his paper; my apology for offering a few remarks on a criticism of a discussion of the subject by myself, which forms a main topic in it, is that I hope to be able to state some definite information which may be worth recording in the American Journal of Mathematics.

What Mr. Basset chiefly deprecates is a method that I have indicated for modifying the theory of magnetic action given originally by Maxwell on the hypothesis of an interacting continuous dynamical system, so as to make it fit in with the ordinary doctrine of reflexion and refraction of light. That theory, as it was applied to the problem of reflexion by Fitz Gerald, does not involve a sufficient number of variables to satisfy the conditions and lead to definite results, such as experiment reveals, independent of accidental conditions at the interface. The theory being dynamical, this implies that some restricting condition in the problem has been overlooked: the variables $\xi$, $\eta$, $\zeta$ are in fact not independent, so that the dynamical variational equation must be subject to the condition that they satisfy, which involves the introduction of a Lagrangian parameter $\lambda$. This Mr. Basset apparently appreciates, and he admits that this new variable obviates the difficulty above mentioned: but he proceeds (loc. cit. §6) to compare this homogeneous theory with Maxwell's equations for an electric field in which there is no rotational influence, and he cannot make out that the one is a sufficiently simple modification of the other. This I think is what the criticism means: but his discussion is based on purely arbitrary interpretations, which

* American Journal of Mathematics, vol. XIX, p. 60.

lead him to the conclusion that the theory is also open to the same objections as was his own. The question is, however, does the theory form a scheme consistent with *itself?* Mr. Basset had already, as he now admits, tried to make the Maxwell equations work, without success: why then should he reject *a priori* without trial the modification of these equations which does work, on the ground that it is not sufficiently slight—for as to logical difficulty there is none? As a matter of fact, the modification is excessively slight in the sense that $\lambda$ would be a very minute quantity that could not be detected in any other way than by experiments on reflexion from magnetized metals. It is complained that there is no definition of the electric force as involving this minute effect: the answer is that there was no necessity to define it, wherein lies a merit of the method. If Mr. Basset so desires, he can run down the analysis of the Maxwellian scheme from the Principle of Least Action, when the rotational property is non-existent, and notice how the quantity usually defined as electric force there comes in: he will thus see that what would correspond to components of electric force, in a rotational medium, are the expressions on the left-hand of his equations (23). He complains that no physical meaning is assigned to $\lambda$: this I suppose indicates a change in his dynamical views, as a few years ago he inserted in the first volume of his Treatise on Hydrodynamics a deduction of the equations of fluid motion from the Principle of Least Action, in which an exactly similar parameter arises by introducing the condition of continuity of flow into the equation of Action. The obvious physical interpretation of the parameter is there the fluid pressure: so here it would represent a hydrostatic pressure, on the hypothesis of an incompressible aether, but if we refrain in Maxwell's manner from particularizing our ideas of the aether we must also refrain from interpreting the parameter.

As to Mr. Basset's modification of his former theory in §10, I agree with him that this is a simpler and in that respect more satisfactory scheme than the one we have just been discussing, because it keeps closer to the spirit of Maxwell's ordinary electrodynamics,—in fact for reasons of molecular theory which I have been intending for some time to set forth in another connexion. But why should Mr. Basset claim it as his special property? In the very paper, a section of which (§§8–11) he is criticising (British Association Report, 1893), he will find it indicated as an alternative method of obtaining a logical scheme of magnetic reflexion. The whole discussion (§§13–20) is of course too long to transcribe: but I quote §20, and take the opportunity to restore the accidental omission of a coefficient, and to give a more conclusive, in fact decisive, reply to the difficulty that is there pointed out at the end.*

"There are thus two ways in which the magnetic field may affect the phenomena of light-propagation. The imposed magnetization is an independent kinetic system of a vortical character which is linked on to the vibrational system which transmits the light waves; the kinetic reaction between the two systems will add on new terms to the electric force: these terms are naturally continuous so long as the medium is continuous, but owing to their foreign origin they need not be continuous at an interface where the magnetized medium suddenly changes. At such an interface the other part of the electric force, which is derived from the vibrating system itself, has been assumed to be continuous in the ordinary manner, viz. its tangential components continuous; the total induction through the interface must of course always maintain continuity. This seems to be the type of theory developed by Maxwell in his hypothesis of molecular vortices ("Treatise," §822), and the conditions to which it leads have been applied to magnetic reflexion by the majority of writers on the subject, including Basset, Drude, J. J. Thomson. But against this procedure there stands the pure assumption as regards discontinuity of electric force at an interface. The correct boundary conditions would be derived from the modification of Fitz Gerald's procedure, which has been explained above.

The other point of view is the purely formal one contemplated by Lord Kelvin and Maxwell in their discussions of possible rotational coefficients introduced into the properties of the medium by magnetization. The magnetization is supposed to slightly alter the structure of the medium which conveys the light-vibrations, but not to exert a direct dynamical effect on these vibrations.

It would appear from the analysis of Drude, and more particularly of J. J. Thomson,† that there is some ground for assuming the correctness of the equations to which the former method leads; and those equations may be

*The section which Mr. Basset quotes as my theory is entitled "Dynamical Theories based on the Form of the Energy-function": the following section, of which the above quotation forms the end, is entitled "Recent Electrical Theories." In a subsequent memoir, which Mr. Basset characterizes, where the aim was to get as much aether theory as possible out of a single energy function of a continuous medium, the theory involving $\lambda$ was the only one that fitted in; but in the appendix to that memoir, and in a second part since published, it was shown that this foundation must be broadened by separate consideration of aether and of matter; and I hope shortly to show in a continuation of the same subject that the hypothesis of molecular constitution therein laid down leads naturally to magnetic rotation of the type of the other theory above described.

†J. J. Thomson, "Recent Advances in Electricity and Magnetism," 1893, §412.

expressed in the terms of the second method somewhat as follows.* The electric current is in a dielectric the rate of change of the electric displacement, which is of an elastic character; in a conducting medium part of the current is due to the continual damping of electric displacement in frictional modes: it may thus fairly be argued that the fundamental relation is primarily not between current and electric force, but between current and displacement, while the current is indirectly expressed in terms of electric force through the elastic relation between displacement and force. The equations would then run as follows, $(\xi, \eta, \zeta)$ being the electric displacement:

$$(u, v, w) = \left(\frac{d}{dt} + \frac{4\pi\sigma}{K}\right)(\xi, \eta, \zeta)$$

where

$$\xi = P - b_3 Q + b_2 R,$$
$$\eta = Q - b_1 R + b_3 P,$$
$$\zeta = R - b_2 P + b_1 Q.$$

This would make the relation between electric displacement and electric force of a rotational character, owing to the magnetization. If the medium were not magnetized, Lord Kelvin's argument might be employed for the negation of such a rotational character on the ground that a sphere rotating in an electric field would generate a perpetual motion; but, as it is, the rotation in the magnetic field would generate other electric forces. The frictional breaking down of displacement, viz. conduction, is known to assume a slightly rotational character, as manifested in the Hall effect."

On this I now make the following remarks. In the equations connecting the electric displacement $(\xi, \eta, \zeta)$ with the electric force $(P, Q, R)$ the coefficient $K/4\pi$ has been accidentally dropped out from the first terms on the right-hand side. On referring back to §15, it will be seen that the coefficients $b_1$, $b_2$, $b_3$ involve $d/dt$, and so are each of the form $\alpha d/dt + \beta$.† Now there is no possibility of this relation admitting perpetual motions, as above suggested, if for transparent media the term $\beta$ vanishes in each coefficient: this I had in that connexion omitted to notice, although it was in fact stated in §16 in a quotation

* "When this transformation is made, Drude's boundary conditions become simply the ordinary ones which express that the tangential components of the electric force and the magnetic force are continuous in crossing the interface; the difficulty as to discontinuity in the tangential electric force does not now occur," *loc. cit.* §18.

† It is to be borne in mind that the general equations are stated throughout for the general case of conducting media.

from Willard Gibbs' general discussion of the subject. Moreover a brief consideration of the case of non-metallic media shows independently (loc. cit. §5) that $\beta$ must vanish, as otherwise the law of dispersion would be entirely different from Biot's law of the inverse square of the wave-length.

Of the two alternative valid magneto-optic schemes thus indicated, the latter leads to the same analysis for the problem of reflexion as Mr. Basset's in Phil. Trans., 1891, and is in fact his present modified scheme. This analysis should also at bottom form a case of the more recent general discussions of Goldhammer and of Drude in their memoirs in Wiedemann's Annalen, 1893, in which the theory is put to the test by means of the actual observations on magnetic reflexion from metals. In this latter connexion I am bound to state that the theory in its valid form as regards boundary conditions was really first formulated by Goldhammer, though without attending to the question of the possibility of perpetual motions (loc. cit. §14); but the unusual character of his notation and analysis had obscured the fact from my notice.*

Although we may form a preference for one or the other of the two schemes from general considerations, the only real test between them is the criterion of agreement with experimental data, which are now voluminous and exact, thanks chiefly to the Dutch physicists. This test is not an easy matter, because the phenomenon is measurable only in reflexion from the magnetic metals, so that the complications of metallic reflexion enter into the analysis. But when the data for such precise verification exist, the mere deduction of formulæ for the simple case of reflexion by transparent media is a minor consideration. I am permitted by Mr. J. G. Leathem, Fellow of St. John's College, Cambridge,† to state that he has worked out a detailed comparison of the theory involving the parameter $\lambda$ with the experimental data of Kundt, Sissingh, and Zeeman, and has found that the agreement is at best very doubtful; also that it is probably better with the second type of theory, with which also in its valid form he became acquainted through my British Association report, and of which his test is not yet complete.‡ As above mentioned, the discussions of Goldhammer and

* I find however that I had introduced all these considerations in a discussion of magneto-optic theory published in the Proceedings of the London Mathematical Society, April, 1893, where Goldhammer's scheme is stated explicitly just as Mr. Basset now gives it, and the negation of perpetual motions is also verified.

† Mr. Leathem has pointed out to me that by an oversight I have incidentally stated that the value of $\lambda$ must be continuous across an interface: this is of course not true, so that $\lambda$ is expressible as the potential of a surface layer *plus* double sheet.

[‡ The complete discussion has fully verified this anticipation. May 1.]

Drude relate to the second type of theory: the latter author takes the rotational constant for metals to be a complex quantity of special form related to the complex refractive index, while the former takes it to be an unrestricted complex quantity. The result of their examination of each other's work in Wiedemann's Annalen seems to be an admission that the experimental determinations employed for purposes of verification were not sufficiently exact to decide between these alternatives. I anticipate that we shall soon have further and perhaps more definite information.

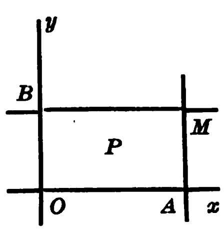
One other point may be noticed, to prevent the misconception that might arise from silence. Mr. Basset remarks (§13) on what he considers to be the fallacy of placing the Principle of Least Action at the foundation of physical dynamics. The question involved is one of deep philosophical interest, and has been a prominent subject of discussion ever since the times of Fermat and Maupertuis. As recent sources of information, reference may be made to a series of papers on the physical meaning and application of the Principle of Least Action, by von Helmholtz (of which the most important is a Memoir in Crelle, vol. 100) republished in the third volume of his Wissenschaftliche Abhandlungen; also to the introduction to Kirchhoff's "Vorlesungen über Mathematische Physik." But possibly Mr. Basset's objection is not serious: for I observe that he has immediately afterwards closely followed my example, by formulating his modified scheme in terms of the Principle of Least Action on the exact lines of Fitz-Gerald and myself. This is, in my view, an important improvement in method, for the founding of the analysis on the Action of Principle has the merit of (to a great extent) avoiding the necessity of separate investigation as to whether perpetual motions are evaded; that it could be simply made I had not noticed, or I should certainly have mentioned the fact in my British Association report of 1893. Although the theory is thus expressed in terms of a magneto-optic energy function, it of course does not follow that it thereby falls under Maxwell's idea of an interaction between two linked dynamical systems: the origin of these energy terms is to be sought in the constitution of the individual molecules, for they involve the continued product of the imposed magnetic field, the electric polarization, and its time rate of change, whereas a direct energy term belonging to the medium treated as continuous would involve the product of only two physical factors.

ST. JOHN'S COLLEGE, CAMBRIDGE, *Jan.* 28, 1897.

# *Exemples d'inversion d'intégrales doubles.*

Par Paul Appell.

Je me propose d'indiquer ici quelques exemples élémentaires du mode d'inversion des intégrales multiples que j'ai énoncé dans une courte Note des Comptes Rendus (1er février 1897). Je me bornerai au cas des variables réelles. L'extension aux variables imaginaires se fera en suivant la voie ouverte par M. Poincaré (Comptes Rendus, 25 janvier, 1886).




I.—Soient $x$ et $y$ deux variables réelles, $F(x, y)$ et $\Phi(x, y)$ deux fonctions *symetriques* de ces deux variables. Nous regarderons $x$ et $y$ comme les coordonnées d'un point $P$ par rapport à deux axes rectangulaires $Ox$ et $Oy$.

Soit $M$ un point de coordonnées

$$\overline{OA} = a, \quad \overline{OB} = b.$$

Considérons les deux équations

$$\left.\begin{aligned} \int\int F(x, y)\,dx\,dy = u, \\ \int\int \Phi(x, y)\,dx\,dy = v, \end{aligned}\right\} \qquad (1)$$

les intégrales doubles étant étendues au rectangle $OAMB$. Ces intégrales sont des fonctions de $a$ et $b$, de sorte que les équations (1) définissent $a$ et $b$ en fonction de $u$ et $v$. On peut se demander ce que doivent être les fonctions $F(x, y)$ et $\Phi(x, y)$ pour que les deux quantités

$$a + b, \quad ab$$

soient des fonctions *uniformes* de $u$ et $v$, et chercher ensuite à étendre cette notion au cas où les variables seraient imaginaires.

Nous nous bornerons ici à donner des exemples simples dans lesquels cette inversion est possible.

1°. Soient les deux équations

$$\left.\begin{aligned} \iint 2(x+y)\,dx\,dy &= u, \\ \iint dx\,dy \qquad &= v. \end{aligned}\right\} \tag{2}$$

En effectuant les intégrations étendues au rectangle *OAMB* on a immédiatement

$$ab(a+b) = u,$$
$$ab = v$$

Donc $a$ et $b$ sont racines de l'équation du deuxième degré

$$X^2 - X\frac{u}{v} + v = 0$$

dont les coefficients sont uniformes en $u$ et $v$.

2°. Soient

$$\left.\begin{aligned} \iint 2(x+y)\,dx\,dy &= u, \\ \iint \frac{(2-x^3y^3)\,dx\,dy}{\sqrt{xy\,(4+x^3y^3)^3}} &= v. \end{aligned}\right\} \tag{3}$$

La première relation donne

$$ab(a+b) = u.$$

La deuxième se transforme comme il suit. Intégrons d'abord par rapport à $y$: nous aurons à calculer

$$dx \int_0^b \frac{(2-x^3y^3)\,dy}{\sqrt{xy\,(4+x^3y^3)^3}};$$

ce qui est égal à

$$dx\,\frac{1}{\sqrt{x}}\,\frac{\sqrt{b}}{\sqrt{4+b^3x^3}}.$$

Il reste ensuite à intégrer par rapport à $x$, ce qui donne

$$\int_0^a \frac{\sqrt{b}\,dx}{\sqrt{x}\sqrt{4+b^3x^3}} = v,$$

ou encore

$$\int_0^a \frac{d\,\frac{1}{bx}}{\sqrt{\frac{4}{(bx)^3}+1}} = -v.$$

On en tire, d'après les notations de Weierstrass

$$\frac{1}{ba} = \wp(v;\ 0,\ -1).$$

Donc, enfin les deux équations (3) sont équivalentes aux deux suivantes

$$\frac{1}{ab} = \wp(v;\ 0,\ -1),$$
$$ab(a+b) = u.$$

Les quantités $a$ et $b$ sont donc racines de l'équation

$$X^2 - u\,\wp(v)\,X + \frac{1}{\wp(v)} = 0$$

à coefficients uniformes en $u$ et $v$.

Au lieu de supposer les intégrales (1) étendues à l'aire du rectangle $OAMB$, on peut les supposer étendues à une aire fermée dont la définition dépend *d'une manière uniforme* de la position du point $M(a, b)$.

Par exemple, on pourrait considérer les deux cercles passant par $O$, par $M$ et ayant respectivement leurs centres sur $Ox$ et $Oy$. On étendrait ensuite les deux intégrales (1) à l'aire $S$ commune à ces cercles. On pourrait également considérer le cercle $C$ décrit sur $OM$ comme diamêtre et étendre les deux intégrales à ce cercle $C$. Prenons ce dernier champ d'intégration $C$ et écrivons les deux équations

$$\int\int dx\,dy = u,$$
$$\int\int (x+y)\,dx\,dy = v.$$

Les coordonnées de $M$ étant $a$ et $b$, ces deux équations donnent

$$\pi(a^2+b^2) = 4u,$$
$$\pi(a^2+b^2)(a+b) = 8v.$$

On en déduit

$$a^2+b^2 = \frac{4u}{\pi},$$

$$a+b = \frac{2v}{u},$$

$$ab = \frac{2v^2}{u^2} - \frac{2u}{\pi}$$

et $a$ et $b$ sont racines de l'équation

$$X^2 - \frac{2v}{u}X + \frac{2v^2}{u^2} - \frac{2u}{\pi} = 0,$$

à coefficients uniformes en $u$ et $v$.

II.—On peut, plus généralement, prendre un champ d'intégration dont la definition dépend d'un nombre quelconque de paramêtres, et écrire des équations d'inversion en nombre égal à celui des paramêtres. Prenons, par exemple, pour champ d'intégration, un cercle quelconque du plan

$$x^2 + y^2 - 2ax - 2by - c \leqq 0.$$

La définition de ce champ dépend de trois variables $a$, $b$, $c$. Considérons les trois équations

$$\left.\begin{aligned} \int\int dx\,dy &= u, \\ \int\int (x+y)\,dx\,dy &= v, \\ \int\int (x^2+y^2)\,dx\,dy &= w. \end{aligned}\right\} \tag{4}$$

D'après la théorie des centres de gravité et des moments d'inertie, on a immédiatement les valeurs de ces trois intégrales étendues au cercle.

Le rayon du cercle étant

$$R = \sqrt{a^2 + b^2 + c},$$

on a les trois équations

$$\begin{aligned} \pi R^2 &= u, \\ \pi R^2 (a+b) &= v, \\ \pi R^2 (a^2+b^2) + \frac{\pi R^4}{2} &= w; \end{aligned}$$

on en tire

$$a + b = \frac{v}{u},$$

$$a^2 + b^2 = \frac{w}{u} - \frac{u}{2},$$

$$a^2 + b^2 + c = \frac{u}{\pi}.$$

Les trois équations (4) définissent donc

$$a + b, \quad a^2 + b^2, \quad c,$$

comme fonctions uniformes de $u$, $v$, $w$.

# *Bemerkungen zu C. S. Peirce Quincuncial Projection.*

Von I. Frischauf, *Graz.*

Im XVIII. Bande dieser Zeitschrift liefert Herr J. Pierpont eine Arbeit über Peirce's Projection. Nachfolgende Bemerkungen mögen als Ergänzung angeführt werden.

Setzt man die Vergrösserungszahl im Mittelpuncte der Karte (Pol) gleich 1, so ist dieselbe für den Punct $(\theta, l)$ der Erdkugel

$$m = \frac{1}{\sqrt{\rho}}, \quad \rho^2 = \sin^2 l + \tfrac{1}{4}\cos^2 l \sin^2 2\theta.$$

In den vier Puncten des Aequators $\theta = 0,\ 90°,\ 180°,\ 270°$ wird $m = \infty$. Wird aben $m$ als eine mässig grosse Zahl vorausgesetzt, so wird $\rho$ klein; es muss daher auch $\sin l$, $\sin l.\theta$ klein sein. Bestimmt man die Fläche $I$ um den Punct $(l = 0,\ \theta = 0)$ herum, wo $\rho \overline{\overline{<}}\, a\ (a < 1)$ ist, so kann dieselbe auf folgende Art bestimmt werden. Sind $s$ und $\alpha$ die sphärischen Polarcoordinaten des Punctes $(l, \theta)$ dieser Fläche, so folgt aus

$$\sin l = \sin s \sin \alpha, \quad \tan \theta = \tan l \cos \alpha,$$
$$\rho^2 = \sin^2 s - \sin^4 s \cos^2 \alpha.$$

Der kleinste Wert von $s$ für einen gegebenen Werth $\rho$ ist $s = \rho$, für $\alpha = 90°$, $270°$; der grösste Werth von $s$ wird für $\alpha = 0,\ 180°$ erhalten aus

$$\sin^2 s - \sin^4 s = \rho^2.$$

Die Fläche $I$, wo $\rho \overline{\overline{<}}\, a$, ist

$$I = \int\int \sin s\, ds\, d\alpha,$$

wo $s$ von 0 bis zum Werthe $s$

$$a^2 = \sin^2 - \delta \sin^4 s \cos^2 \alpha \qquad (1)$$

und $\alpha$ von 0 bis $2\pi$ zu nehmen ist.

Die Integration nach $s$ liefert

$$I = \int_0^{2\pi} (1 - \cos s)\, da\,;$$

aus (1) folgt

$$\cos^2 s = 1 - a^2 - a^4 \cos^2 \alpha - 2a^6 \cos^4 \alpha \ldots\ldots,$$
$$\cos s = 1 - \tfrac{1}{2} a^2 - (\tfrac{1}{2} \cos^2 \alpha + \tfrac{1}{8})\, a^4 - (\cos^4 \alpha + \tfrac{1}{4} \cos^2 \alpha + \tfrac{1}{16})\, a^6 \ldots\ldots;$$

damit $$I = (a^2 + \tfrac{3}{4} a^4 + \tfrac{9}{8} a^6 + \ldots\ldots)\,\pi.$$

Die Fläche auf der ganzen Kugel, wo $\rho \overline{\gtreqless} a$ ist, ist das Vierfache von $I$, das Verhältniss $4I$ zur Kugelfläche $= 4\pi$ ist daher

$$\frac{I}{\pi} = a^2 + \tfrac{3}{4} a^4 + \tfrac{9}{8} a^6 + \ldots\ldots$$

Für $m = 2$ wird $a^2 = \frac{1}{16}$, also

$$\frac{I}{\pi} = 0.0657 = 6.57 \text{ per cent.}$$

Das "Jahrbuch der Fortschritte der Mathematik" (Band XI) gibt daher unrichtig an, dass diese Oberfläche "nur 9 per cent ausmache."

Schliesslich mag noch bemerkt werden, dass die Curvensysteme, welche durch die elliptischen Functionen ausgedrückt werden, bereits von Siebeck (Crelle, Journal für Mathematik, Band 57) behandelt sind.

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## BERICHTIGUNG.

Auf p. 12, Zeile 12 v. o. meiner Abhandlung Vol. XIX, ist zu lesen

$$3n - \sum y_u$$

an Stelle von $n^3 - \sum x^3 + \sum y_u^3$.

S. KANTOR.

# *On the Sign of a Determinant's Term.*

By Ellery W. Davis.

Let $k$ denote the row number and $l$ the column number of an element $(kl)$ in a determinant.

If in the determinant we interchange the $k^{th}$ and $p^{th}$ rows, then in the new determinant the elements that were called $(kl)$ and $(pq)$ in the old should be called $(pl)$ and $(kq)$. All other elements in any term to which $(kl)$ and $(pl)$ belong would remain unchanged in place and therefore in name.

Establish now a one-to-one correspondence between the elements $(k, l)$ and the tracts of the lines $x/k + y/k = 1$ that lie between the axes of coordinates. Call the tracts $(k, l)$.

Then I say that the sign of a term of the determinant is positive or negative according as these tracts cross one another an even or an odd number of times.

That it is so for the term given by the principal diagonal is plain. It remains to show that interchanging two rows of the determinant will alter the parity of the number of crossings.

Let $(k, l)$ cross $(p, q)$ and interchange the $k^{th}$ and $p^{th}$ rows, getting $(p, l)$ and $(k, q)$. On drawing a simple diagram, it will become plain that:

i. $(p, l)$ will not cross $(k, q)$.

ii. Any tract crossed by $(k, l)$ but not by $(p, q)$ will be crossed by $(p, l)$ but not by $(k, q)$, and that likewise any tract crossed by $(p, q)$ but not by $(k, l)$ will be crossed by $(k, q)$ but not by $(p, l)$.

iii. Any tract crossed by both $(k, l)$ and $(p, q)$, or by neither $(k, l)$ nor $(p, q)$, will be crossed by both or neither of the tracts $(p, l)$ and $(k, q)$.

Now ii. does not change the number of crossings, iii. does not change the parity of the number, i. does change the parity. Therefore, upon the whole, the parity is changed and our theorem is proved.

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